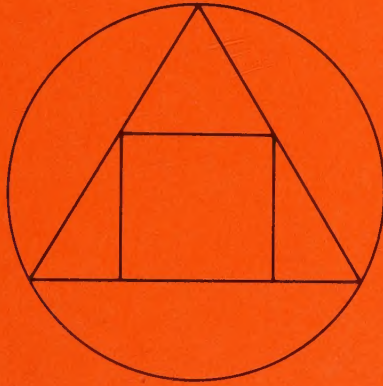


Transactions of

MACHINE ELEMENTS

DIVISION

Lund Technical University
Lund , Sweden



On the Design of External Involute
Helical Gears

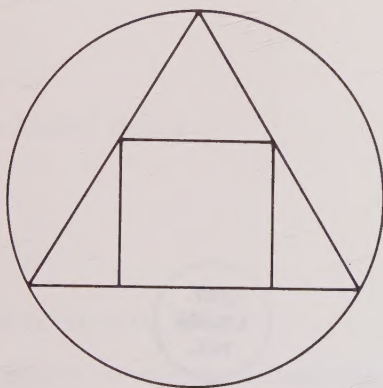
Lars Vedmar
Lund 1981

Transactions of

MACHINE ELEMENTS

DIVISION

Lund Technical University
Lund , Sweden



On the Design of External Involute
Helical Gears

Lars Vedmar
Lund 1981

Acknowledgements

This work was carried out at the Machine Elements Division, Lund Technical University, Lund, Sweden, under the supervision of the Head of the Division, Professor Leif Floberg.

I wish to express my gratitude to Karl Hellborg and Torsten Jakobsson who built the test apparatus. I also gratefully acknowledge the assistance of Rose-Marie Hermansson who typed the report and Birgitta Tell who prepared all illustrations.

Lars Vedmar



ON THE DESIGN OF EXTERNAL INVOLUTE
HELICAL GEARS

Av

Lars Vedmar

Civilingenjör, Malmö nation

AKADEMISK AVHANDLING

som för avläggande av teknisk doktorsexamen vid tekniska fakulteten vid Universitetet i Lund kommer att offentligen försvaras å hörsal M:B, Maskinteknik, Ole Römers väg 1, Lund fredagen den 22 maj 1981 kl. 9.15.

Organization LUND UNIVERSITY		Document name DOCTORAL DISSERTATION	
Department of Machine Elements Box 725 220 07 LUND		Date of issue 810428	
Author(s) Lars Vedmar		CODEN: LUTMDN/(TMME-1001)/1-100/(1981)	
Sponsoring organization		Title and subtitle On the Design of External Involute Helical Gears	
Abstract A theoretical investigation of the stresses in external involute helical gears has been carried out. Geometrical conditions are given and the load distribution along and between the gear teeth in contact is calculated. With this load distribution, contact pressure and fillet tensile and compression stresses are calculated. The influence bending functions, used to calculate the load distribution, and the fillet stresses are calculated by the finite element method. Results are presented for the contact pressure and fillet stresses as functions of the number of teeth and the helix angle.			
Key words Contact stress, Fillet stress, Gear strength, Helical gear, Load distribution, Tooth Pitting.			
Classification system and/or index terms (if any)			
Supplementary bibliographical information			Language English
ISSN and key title			ISBN
Recipient's notes		Number of pages 100	Price —
Security classification		Distribution by (name and address): Department of Machine Elements, Box 725, 220 07 LUND	

I, the undersigned, being the copyright owner of the abstract of the above-mentioned dissertation, hereby grant to all reference sources permission to publish and disseminate the abstract of the above-mentioned dissertation

Signature Lars Vedmar

Date 810402

Contents

Acknowledgements	2
1. Introduction	4
2. Notation	5
3. Geometry of External Involute Gears	7
3.1. Gear Profile Generated by a Rack Tool	9
3.2. Geometry of a Gear with Tip Relief	15
3.3. Gear Meshing Conditions	19
3.4. Gear Existence Conditions	21
4. Load Distribution	25
4.1. Contact Deformation	26
4.2. Bending Deformation	27
4.3. Condition of Deformation	30
4.4. Load Distribution Solution	31
5. Gear Contact Pressures	44
6. Gear Bending Stresses	60
7. Experimental Investigation	78
8. Conclusion	82
9. Appendices	83
A1. Three-dimensional Study of the Contact Deformation	83
A2. Influence Function of Bending Deformation	88
A3. Tables of Calculated Values	92
10. References	100

1. Introduction

Gear strength is divided into two parts, namely contact pressures and fillet bending stresses.

The contact pressures are dealt with by referring to the problem of two elastic bodies in contact, first treated by Hertz.

In the spur gear case the fillet stresses were first treated with the elementary cantilever bending theory, and then completed with stress concentration factors, based upon photoelastic tests. The first really important treatment of bending stresses was carried out by Lewis (5) as early as 1893. To calculate the fillet stresses in helical gears the gears have been approximately transformed to spur gears using a virtual number of teeth, as for example by Buckingham (2) and Niemann (6).

In earlier works Wennerström (9) and (10) made calculations for external spur gears considering geometry, contact pressures and fillet stresses and for external helical gears considering contact pressures, all under the assumption that the load is uniformly distributed over the whole contact length. Corresponding calculations for internal spur gears have been made by Andersson (1).

When calculating contact pressures and fillet stresses the load distribution has to be known. Many authors have calculated the load distribution in helical gears using influence bending functions obtained from experiments or bending of cantilever plates. Here, reference is made to Hayashi and Sayama (3).

The aim of this work is to determine the load distributions in external involute helical gears and, using these, to calculate contact pressures and fillet stresses. The influence function of bending deformation and the bending stresses are treated with the finite element method.

Diagrams are made for the contact pressures and the fillet stresses as functions of the number of teeth at the pinion and the gear ratio when the helix angle is 0, 10° and 20° respectively.

2. Notation

A	Constant, point
a	Centre distance
B	Constant
$b (b_0 = b/m_n)$	Gear face width (non-dimensional)
C	Constant
$c (c_0 = c/m_n)$	Bottom clearance (non-dimensional)
E	Modulus of elasticity
e	Displacement
$F (F_0 = \frac{F}{E_p m_n^2})$	Point, force (non-dimensional)
g	Radius of base cylinder
H	Point
$h (h_0 = h/m_n)$	Height, distance (non-dimensional)
I	Point
i, j	Number
K	Number of simultaneous contact lines
k	Number, material constant
$L (L_0 = L/m_n)$	Length (non-dimensional)
l	Number
M	Torque
m	Gear module
n	Number
$\hat{n}$	Unit vector
$p (p_0 = p \sqrt{\frac{a^2 b (k_p + k_g)}{M_p}})$	Contact pressure (non-dimensional)
Q	Load
$q (q_0 = q m_n/F)$	Load intensity (non-dimensional)
R	Point
$r (r_0 = r/m_n)$	Radius (non-dimensional)
$\hat{s}$	Unit vector
T	Point
t	Transverse base pitch
$\hat{t}$	Unit vector
U	Gear ratio

$u (u_0 = u E_p m_n / F)$	Displacement (non-dimensional)
x	Modification factor, coordinate
y	Coordinate
z	Number of teeth, coordinate
$\alpha (\alpha_0 = \alpha E_p m_n)$	Pressure angle, influence function of bending deformation (non-dimensional)
β	Helix angle
Γ	Function
γ	Lead angle
Δ	Constant
δ	Distance
ϵ	Gear contact ratio, small number, strain component
ξ, η	Coordinate
θ	Base cylinder helix angle
κ	Angle
λ	Angle
ν	Poisson's ratio
ξ	Coordinate
ρ	Radius of curvature
$\sigma (\sigma_0 = \sigma a^2 b / M_p)$	Normal stress (non-dimensional)
ϕ	Function
φ	Angle
ψ	Function

Indices

0	Non-dimensional
f	Fillet
g	Gear
i	Involute, number
j, k, l	Number
n	Normal
p	Pinion
t	Tool

3. Geometry of External Involute Gears

One of the main properties of involute gears is that the gear becomes a rack, with straightlined flanks in a plane perpendicular to the axis of rotation of the co-operating gear, when the number of teeth is approaching infinity.

This rack can co-operate with every other involute gear, provided the module and the helix angle are the same, with straightlined contact lines when, and only when, the flanks of the rack are plane.

These properties make it possible to manufacture involute gears working together with straightlined contact lines, using a tool shaped like a rack with plane flanks.

Fig. 3.1 shows such a rack with the normal pressure angle α_n , the helix angle β_n , the tool height above the median line $h_{0t}m_n$ and the tool tip radius $r_{0t}m_n$.

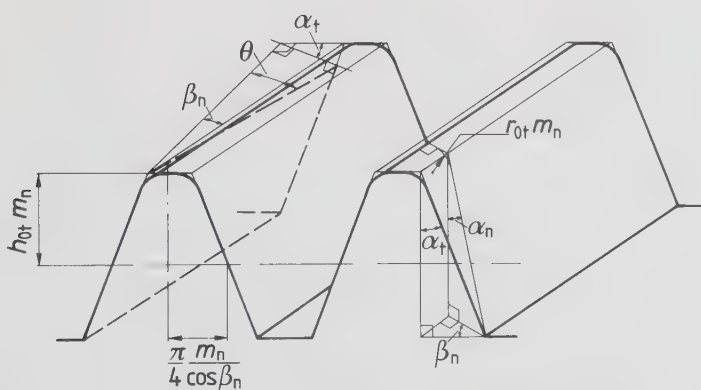


Fig. 3.1. Geometry of rack tool

Throughout this work the values $\alpha_n = 20^\circ$, $h_{0t} = 1.25$ and $r_{0t} = 0.35$ are used. The pressure angle during the rack tool generation in the transverse plane

α_t , and the base cylinder helix angle θ can be determined from Fig. 3.1 where

$$\tan \alpha_t = \frac{\tan \alpha_n}{\cos \beta_n} \quad (3.1)$$

and

$$\tan \theta = \tan \beta_n \cos \alpha_t$$

Substituting the last equation in Eq. 3.1 we find

$$\tan \theta = \frac{\sin \beta_n}{\sqrt{\cos^2 \beta_n + \tan^2 \alpha_n}} \quad (3.2)$$

The pitch in the transverse plane is

$$\pi m_t = \pi \frac{m_n}{\cos \beta_n} = \frac{2\pi r}{z}$$

where r is the pitch radius and z the number of teeth of the gear co-operating with the tool. From this equation we get the gear module of the rack tool in the transverse plane

$$m_t = \frac{m_n}{\cos \beta_n} \quad (3.3)$$

and the pitch radius

$$r = \frac{m_n z}{2} \frac{1}{\cos \beta_n} \quad (3.4)$$

The base circle radius

$$g = r \cos \alpha_t = \frac{m_n z}{2} \frac{1}{\sqrt{\cos^2 \beta_n + \tan^2 \alpha_n}} \quad (3.5)$$

Since every point of the gear in a transverse plane $\xi = \text{constant}$ remains in this plane during the whole manufacturing process, it is sufficient to study the plane where the geometry is to be determined, in this case the plane $\xi = 0$.

To the right in Fig. 3.2 the tool is also shown in its normal plane, i.e. the tool twisted the angle β_n with respect to the generating transverse plane of the tool.

The coordinate system (ξ, η, ζ) is fixed to the side of the gear blank, at the axis of rotation, and the coordinates are divided by the rack tool normal module m_n , thus becoming non-dimensional.

The straightlined flank of the tool cuts the gear involute flank at a point I while the plane tip of the tool cuts the gear foot circle. The tip radius of the tool in a normal plane, transformed to an ellipse in a transverse plane, generates a curve connecting the involute curve and the foot circle called the fillet curve or just the fillet. In Fig. 3.2 the tool cuts the fillet at a point F .

When $\varphi = 0$ the common point of the fillet and the foot circle is cut, and in this position of the tool the points R and R_1 are identical.

From Fig. 3.2 we obtain when $\overline{OR} = z/(2 \cos \beta_n)$

$$\begin{aligned} \overline{HR} &= \overline{HR_1} + \overline{R_1 R} = \left[h_{0t} - x_n - r_{0t} (1 - \sin \alpha_n) \right] \tan \alpha_t + r_{0t} \frac{\cos \alpha_n}{\cos \beta_n} + \\ &+ \frac{z}{2} \frac{\varphi}{\cos \beta_n} = \left[h_{0t} - x_n + r_{0t} \frac{1 - \sin \alpha_n}{\sin \alpha_n} \right] \tan \alpha_t + \frac{z}{2} \frac{\varphi}{\cos \beta_n} \quad (3.6) \end{aligned}$$

and

$$\overline{FR} = \frac{h_{0t} - x_n - r_{0t} (1 - \sin \lambda_n)}{\sin \lambda} \quad (3.7)$$

and

$$\begin{aligned} \varphi_1 &= \frac{\frac{\pi}{4} \frac{1}{\cos \beta_n} + \overline{HR_1} + x_n \tan \alpha_t}{\frac{z}{2} \frac{1}{\cos \beta_n}} = \\ &= \frac{\frac{\pi}{4} \frac{1}{\cos \beta_n} + x_n \tan \alpha_t + \left[h_{0t} - x_n + r_{0t} \frac{1 - \sin \alpha_n}{\sin \alpha_n} \right] \tan \alpha_t}{\frac{z}{2} \frac{1}{\cos \beta_n}} = \end{aligned}$$

$$= \frac{2}{z} \left[\frac{\pi}{4} + h_{0t} \tan \alpha_n + r_{0t} \frac{1 - \sin \alpha_n}{\cos \alpha_n} \right] \quad (3.8)$$

To find the connection between the angles λ and φ we need

$$\tan \lambda = \frac{\tan \lambda_n}{\cos \beta_n} = \frac{h_{0t} - x_n - r_{0t} (1 - \sin \lambda_n)}{r_{0t} \frac{\cos \lambda_n}{\cos \beta_n} + \frac{z}{2} \frac{\varphi}{\cos \beta_n}}$$

giving

$$\varphi = \frac{h_{0t} - x_n - r_{0t} \left[1 - \sin \lambda_n \left(1 - \frac{1}{\cos^2 \beta_n} \right) \right]}{\frac{z}{2} \tan \lambda_n} \cos^2 \beta_n =$$

$$= \frac{2 \cos^2 \beta_n}{z \tan \lambda_n} [h_{0t} - x_n - r_{0t} (1 + \sin \lambda_n \tan^2 \beta_n)] \quad (3.9)$$

The non-dimensional coordinates $\xi_{i, \zeta=0}$ and $\eta_{i, \zeta=0}$ of the involute flank, expressed in the parameter φ , are

$$\left\{ \begin{aligned} \xi_{i, \zeta=0} &= \frac{z}{2} \frac{1}{\cos \beta_n} \sin (\varphi_1 + \varphi) - \left[\left[h_{0t} - x_n + r_{0t} \frac{1 - \sin \alpha_n}{\sin \alpha_n} \right] \tan \alpha_t + \right. \\ &\quad \left. + \frac{z}{2} \frac{\varphi}{\cos \beta_n} \right] \cos \alpha_t \cos (\varphi_1 + \varphi - \alpha_t) \\ \eta_{i, \zeta=0} &= \frac{z}{2} \frac{1}{\cos \beta_n} \cos (\varphi_1 + \varphi) + \left[\left[h_{0t} - x_n + r_{0t} \frac{1 - \sin \alpha_n}{\sin \alpha_n} \right] \tan \alpha_t + \right. \\ &\quad \left. + \frac{z}{2} \frac{\varphi}{\cos \beta_n} \right] \cos \alpha_t \sin (\varphi_1 + \varphi - \alpha_t) \end{aligned} \right. \quad (3.10)$$

and the non-dimensional coordinates $\xi_{f, \zeta=0}$ and $\eta_{f, \zeta=0}$ of the fillet are

$$\left\{ \begin{aligned} \xi_{f, \zeta=0} &= \frac{z}{2} \frac{1}{\cos \beta_n} \sin (\varphi_1 + \varphi) - \frac{h_{0t} - x_n - r_{0t} (1 - \sin \lambda_n)}{\sin \lambda} \cos (\varphi_1 + \varphi - \lambda) \\ \eta_{f, \zeta=0} &= \frac{z}{2} \frac{1}{\cos \beta_n} \cos (\varphi_1 + \varphi) + \frac{h_{0t} - x_n - r_{0t} (1 - \sin \lambda_n)}{\sin \lambda} \sin (\varphi_1 + \varphi - \lambda) \end{aligned} \right. \quad (3.11)$$

The common point of the involute and the fillet curves can be determined from Eqs. 3.10 and 3.11 for the same angle $\varphi = \varphi_{if}$, provided the gear tooth is not undercut. Substituting $\lambda = \alpha_t$ in Eq. 3.9 we find

$$\varphi_{if} = \frac{2}{z} \frac{\cos^2 \beta_n}{\tan \alpha_n} [h_{0t} - x_n - r_{0t} (1 + \sin \alpha_n \tan^2 \beta_n)] \quad (3.12)$$

If the gear tooth is not undercut the fillet curve can be described by Eq. 3.11 where $0 \leq \varphi \leq \varphi_{if}$.

It is convenient to be able to directly determine the coordinates of the involute curve in a certain point at the contact line. For this reason we need a function $\varphi = \varphi(r_i)$, where r_i is the radius of the current involute point. Using the cosine theorem this radius can be written

$$r_{0i}^2 = \left(\frac{r_i}{m_n} \right)^2 = \overline{HR}^2 \cos^2 \alpha_t + \overline{OR}^2 - 2 \overline{HR} \overline{OR} \cos \alpha_t \cos \left(\frac{\pi}{2} - \alpha_t \right)$$

Substituting Eq. 3.6 in this equation we get after some calculations

$$\varphi = \left[\frac{2}{z} \cos \beta_n \left(\frac{z}{2} \frac{1}{\cos \beta_n} - h_{0t} + x_n - r_{0t} \frac{1 - \sin \alpha_n}{\sin \alpha_n} \right) \pm \frac{\sqrt{\left(r_{0i} \frac{2}{z} \cos \beta_n \right)^2 - \cos^2 \alpha_t}}{\sin \alpha_t} \right] \tan \alpha_t \quad (3.13)$$

The square root becomes zero when $r_i = g$. The plus sign of the square root gives φ valid for the fillet curve when the tooth is undercut, but is never valid for the involute curve.

The radius r_{if} of the common point of the involute and the fillet curves we obtain by substituting Eq. 3.12 in Eq. 3.13, provided the tooth is not undercut.

After some calculations we obtain

$$r_{0if} = \sqrt{\frac{(z/2)^2}{\cos^2 \beta_n + \tan^2 \alpha_n} + \left[\frac{\tan \alpha_n}{\cos \beta_n} \frac{z/2}{\sqrt{\cos^2 \beta_n + \tan^2 \alpha_n}} - \frac{\sqrt{\cos^2 \beta_n + \tan^2 \alpha_n}}{\tan \alpha_n} [h_{0t} - x_n - r_{0t} (1 - \sin \alpha_n)] \right]^2} \quad (3.14)$$

The next step is to determine the gear geometry in a transverse plane $\zeta \neq 0$, but since the geometry of the tool is the same in every transverse plane the gear geometry, with respect to the symmetry axis of the tooth in a transverse plane, is independent of the position of this plane. However, the gear tooth must be twisted an angle $\gamma\zeta$ to reach the same position with respect to the tool as when $\zeta = 0$.

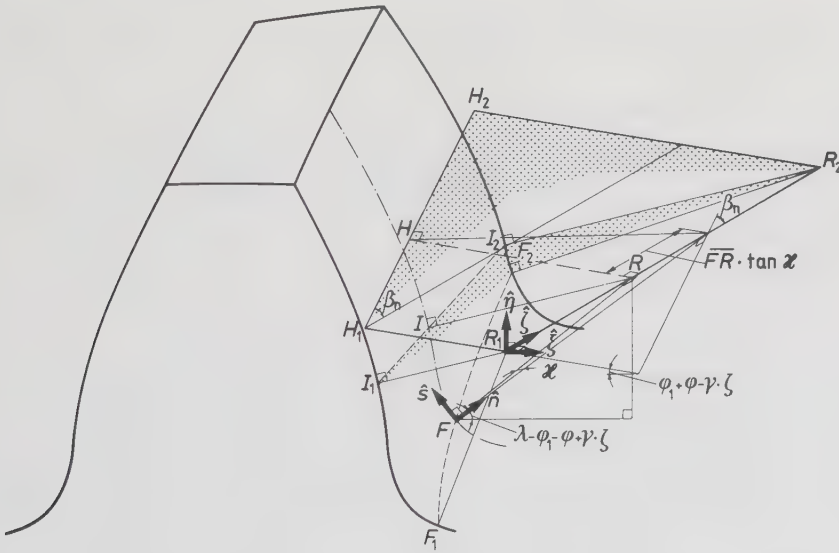


Fig. 3.3. Gear tooth generating in space

Fig. 3.3 shows, schematically, the tool in a position when the angle between the ξ -axis and the median line of the tool is $\varphi_1 + \varphi - \gamma\zeta$, where φ defines the position of the tool when the point with the coordinates $\xi_{i,\zeta=0}$ and $\eta_{i,\zeta=0}$ was cut. In the position shown the tool cuts, among other points at the contact line $\overline{I_1 I_2}$, the point I with the coordinates (ξ_i, η_i, ζ_i) where $\xi_i^2 + \eta_i^2 = \xi_{i,\zeta=0}^2 + \eta_{i,\zeta=0}^2$.

Between these two positions the tool has been rolling the arc length $\overline{HR} - \overline{H_1 R_1}$ on the pitch circle. This means that the symmetry axis of the tooth in the plane $\zeta \neq 0$ is twisted an angle $\gamma\zeta$ with respect to the symmetry axis of the tooth in the plane $\zeta = 0$ where

$$\gamma = \frac{\overline{HR} - \overline{H_1 R_1}}{\frac{z}{2} \cdot \frac{\zeta}{\cos \beta_n}} = \frac{\zeta \tan \beta_n}{\frac{z}{2} \cdot \frac{\zeta}{\cos \beta_n}} = \frac{2 \sin \beta_n}{z} \quad (3.15)$$

If one point at $\zeta = 0$ with the coordinates $\xi_{\zeta=0}$ and $\eta_{\zeta=0}$ has been determined, corresponding coordinates at the same radius in the plane $\zeta \neq 0$ are

$$\begin{cases} \xi = \xi_{\zeta=0} \cos(\gamma\zeta) - \eta_{\zeta=0} \sin(\gamma\zeta) \\ \eta = \xi_{\zeta=0} \sin(\gamma\zeta) + \eta_{\zeta=0} \cos(\gamma\zeta) \end{cases} \quad (3.16)$$

Fig. 3.3 can also be used to determine the direction of the fillet surface normal. If the coordinate system $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ is twisted so that $\hat{\xi}$ becomes the normal direction of the fillet at $\zeta = \overline{R_1 R}$ and $\hat{\eta}$ remains in the transverse plane, we obtain the coordinate system $(\hat{n}, \hat{s}, \hat{t})$. Since $\overline{FR}$ is perpendicular to the fillet in the transverse plane we obtain

$$\hat{n} = \frac{(\cos(\lambda - \varphi_1 - \varphi + \gamma\zeta), \sin(\lambda - \varphi_1 - \varphi + \gamma\zeta), \tan \kappa)}{\sqrt{1 + \tan^2 \kappa}}$$

where κ is the angle between $\hat{n}$ and $\overline{FR}$.

$\hat{n}$ must be perpendicular to the surface of the tool in the point F and the tangential direction of this surface is

$$\overline{H_1 H} = (-\sin \beta_n \cos(\varphi_1 + \varphi - \gamma\zeta), \sin \beta_n \sin(\varphi_1 + \varphi - \gamma\zeta), \cos \beta_n)$$

The scalar product of this vector and $\hat{n} \sqrt{1 + \tan^2 \kappa}$ is

$$\begin{aligned} \sin \beta_n [-\cos(\varphi_1 + \varphi - \gamma\zeta) \cos(\lambda - \varphi_1 - \varphi + \gamma\zeta) + \sin(\varphi_1 + \varphi - \gamma\zeta) \sin(\lambda - \varphi_1 - \varphi + \gamma\zeta)] + \\ + \cos \beta_n \tan \kappa = -\sin \beta_n \cos \lambda + \cos \beta_n \tan \kappa = 0 \end{aligned}$$

giving

$$\tan \kappa = \tan \beta_n \cos \lambda$$

The directions of the fillet are now

$$\left\{ \begin{array}{l} \hat{n}_f = \frac{(\cos (\lambda - \varphi_1 - \varphi + \gamma \xi), \sin (\lambda - \varphi_1 - \varphi + \gamma \xi), \tan \beta_n \cos \lambda)}{\sqrt{1 + \tan^2 \beta_n \cos^2 \lambda}} \\ \hat{s}_f = (-\sin (\lambda - \varphi_1 - \varphi + \gamma \xi), \cos (\lambda - \varphi_1 - \varphi + \gamma \xi), 0) \\ \hat{t}_f = \frac{(-\tan \beta_n \cos \lambda \cos (\lambda - \varphi_1 - \varphi + \gamma \xi), -\tan \beta_n \cos \lambda \sin (\lambda - \varphi_1 - \varphi + \gamma \xi), 1)}{\sqrt{1 + \tan^2 \beta_n \cos^2 \lambda}} \end{array} \right. \quad (3.17)$$

By setting $\lambda = \alpha_t$ we obtain the directions of the involute surface and specially

$$\hat{n}_i = \frac{(\cos (\alpha_t - \varphi_1 - \varphi + \gamma \xi), \sin (\alpha_t - \varphi_1 - \varphi + \gamma \xi), \tan \beta_n \cos \alpha_t)}{\sqrt{1 + \tan^2 \beta_n \cos^2 \alpha_t}} \quad (3.18)$$

3.2. Geometry of a Gear with Tip Relief

On account of the high contact pressures in the outermost contact points, and to reduce the dynamic oscillations, it is common practice to manufacture the gears with tip relief.

The purpose of this section is to determine the distance δ between a point of the imaginary involute flank I , defined by the coordinates ξ_i and η_i , and the flank with tip relief, defined by the coordinates ξ and η , perpendicular to the involute flank in the transverse plane.

Fig. 3.4 shows the tool in its normal and transverse planes in a position where the point T is cut, where the tip relief is received by the arc defined by the lengths δ_1 and δ_2 in the normal plane of the tool and all lengths are divided by the rack tool normal module m_n , thus becoming non-dimensional.

First we need the radius ρ_t of the tool in its normal plane, expressed in the parameters δ_1 and δ_2 . From Fig. 3.4 we obtain

$$\left\{ \begin{array}{l} \delta_1 + \delta_2 \tan \alpha_n = \rho_t (\cos \alpha_n - \cos \lambda_{n, \max}) \\ \delta_2 = \rho_t (\sin \lambda_{n, \max} - \sin \alpha_n) \end{array} \right.$$

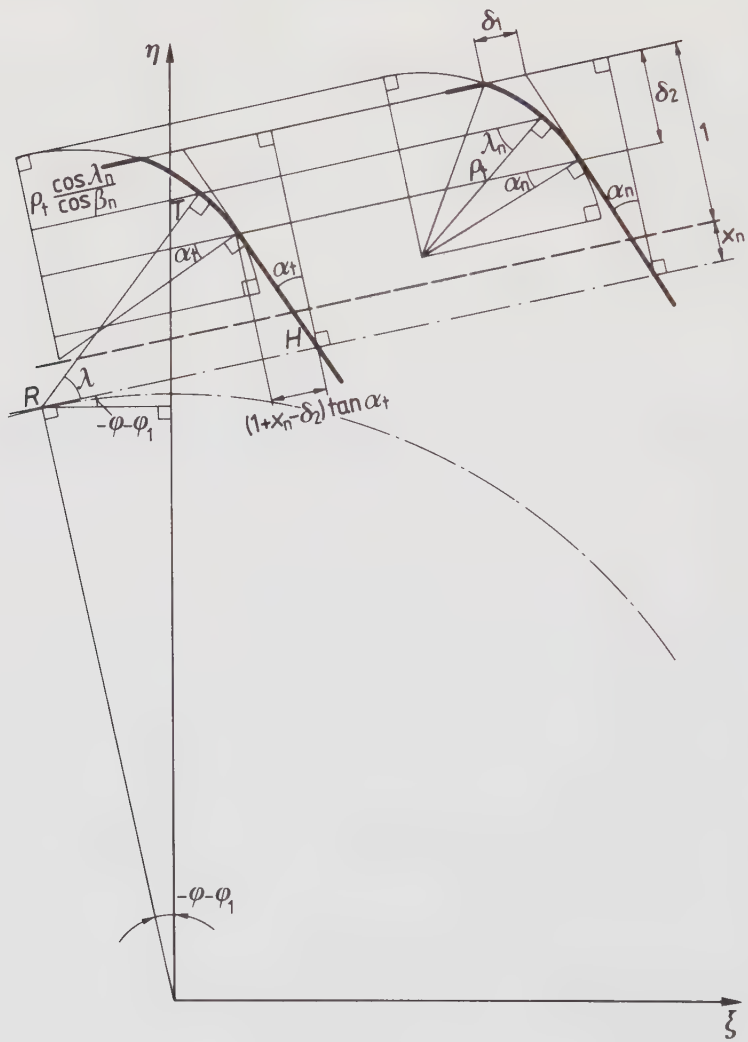


Fig. 3.4. Tip relief generation

Using the latter equation for eliminating $\lambda_{n, \max}$ we obtain after some calculations

$$\rho_t = \frac{(\delta_1 + \delta_2 \tan \alpha_n)^2 + \delta_2^2}{2 \cdot \delta_1 \cos \alpha_n} \quad (3.19)$$

The next step is to determine the coordinates ξ and η of the tooth in the point T . From Fig. 3.4 we obtain

$$\overline{RT} = \frac{1 + x_n - \delta_2 + \rho_t (\sin \lambda_n - \sin \alpha_n)}{\sin \lambda}$$

and

$$\tan \lambda = \frac{\tan \lambda_n}{\cos \beta_n} = \frac{1 + x_n - \delta_2 + \rho_t (\sin \lambda_n - \sin \alpha_n)}{\overline{RH} - (1 + x_n - \delta_2) \tan \alpha_t - \frac{\rho_t}{\cos \beta_n} (\cos \alpha_n - \cos \lambda_n)} \quad (3.20)$$

$\overline{HR} = -\overline{RH}$ from Eq. 3.6 inserted in Eq. 3.20 gives

$$\begin{aligned} -\varphi \frac{z}{2} = & \left[h_{0t} - x_n + r_{0t} \frac{1 - \sin \alpha_n}{\sin \alpha_n} \right] \tan \alpha_n + \\ & + \rho_t \left[\cos \alpha_n (1 - \cos^2 \beta_n \frac{\tan \alpha_n}{\tan \lambda_n}) - \cos \lambda_n \sin^2 \beta_n \right] + \\ & + (1 + x_n - \delta_2) \left(\tan \alpha_n + \frac{\cos^2 \beta_n}{\tan \lambda_n} \right) \end{aligned} \quad (3.21)$$

giving the connection between the angles λ and φ .

The coordinates of the point T are

$$\begin{cases} \xi = \overline{OR} \sin (\varphi_1 + \varphi) - (-\overline{RT}) \cos (\varphi_1 + \varphi - \lambda) \\ \eta = \overline{OR} \cos (\varphi_1 + \varphi) + (-\overline{RT}) \sin (\varphi_1 + \varphi - \lambda) \end{cases} \quad (3.22)$$

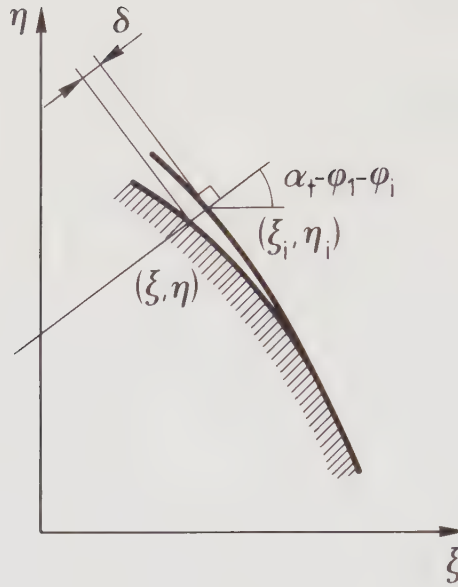


Fig. 3.5. Tip relief geometry

The coordinates of the involute point I from Eqs. 3.6 and 3.10 and from Fig. 3.5 are

$$\begin{cases} \xi_i = \overline{OR} \sin(\varphi_1 + \varphi_i) - \overline{HR} \cos \alpha_t \cos(\varphi_1 + \varphi_i - \alpha_t) = \xi + \delta \cos(\alpha_t - \varphi_1 - \varphi_i) \\ \eta_i = \overline{OR} \cos(\varphi_1 + \varphi_i) + \overline{HR} \cos \alpha_t \sin(\varphi_1 + \varphi_i - \alpha_t) = \eta + \delta \sin(\alpha_t - \varphi_1 - \varphi_i) \end{cases} \quad (3.23)$$

where $\alpha_t - \varphi_1 - \varphi_i$ is the direction of the involute normal according to Eq. 3.18.

By eliminating δ in Eq. 3.23 we obtain after some calculations

$$\alpha_t - \varphi_1 - \varphi_i = \arccos \frac{\overline{OR} \cos \alpha_t}{\sqrt{\xi^2 + \eta^2}} - \arctan \frac{\xi}{\eta} \quad (3.24)$$

which, substituted in Eq. 3.10, gives the coordinates ξ_i and η_i .

When the coordinates of the involute and the relief curves are known the distance δ can be determined from Eq. 3.23 for the position defined by the radius $\sqrt{\xi_i^2 + \eta_i^2}$, which is independent of the axial coordinate ξ .

In the continuation, when the gears are relieved, $\delta_2 = 0.5$ is used.

An example of δ as a function of a coordinate along the path of contact is shown in Fig. 3.6 for some values of δ_1 .

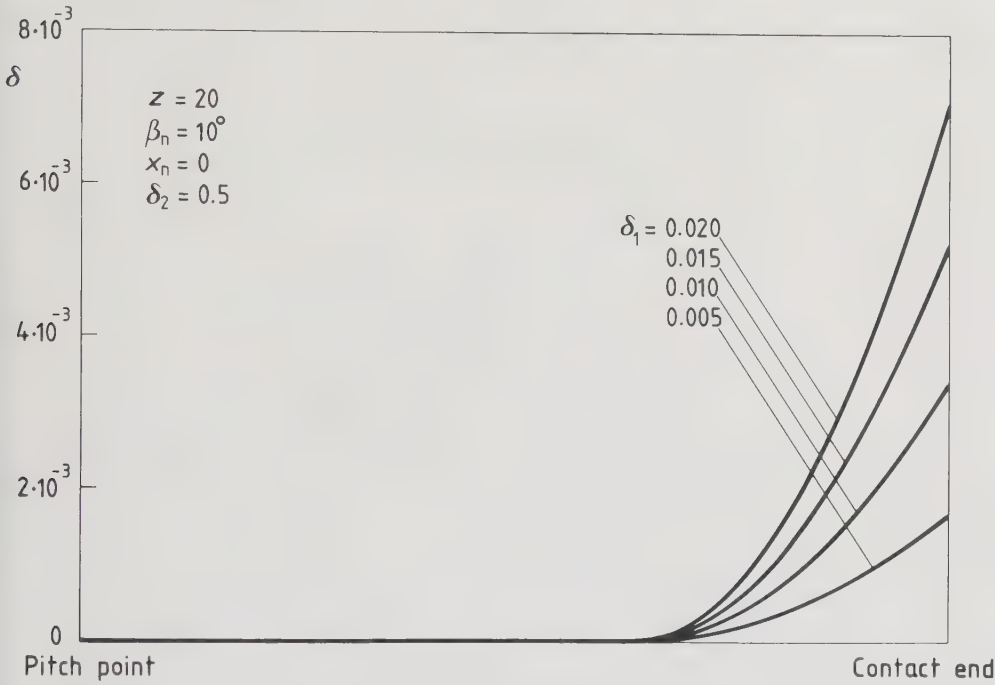


Fig. 3.6. Tip relief backlash

3.3. Gear Meshing Conditions

Now we are going to define the geometrical properties of two external involute helical gears working together, both manufactured in the way previously described but without tip relief, where the smaller one is called the pinion while the larger one is called the gear.

Fig. 3.7 shows the gear meshing region where the points A_p and A_g are the outermost points of the pinion and the gear.

The tip radii of the gears are adjusted to obtain the standard tip-bottom clearance

$$c_t = c_{0t} m_n = (h_{0t} - 1) m_n \quad (3.25)$$

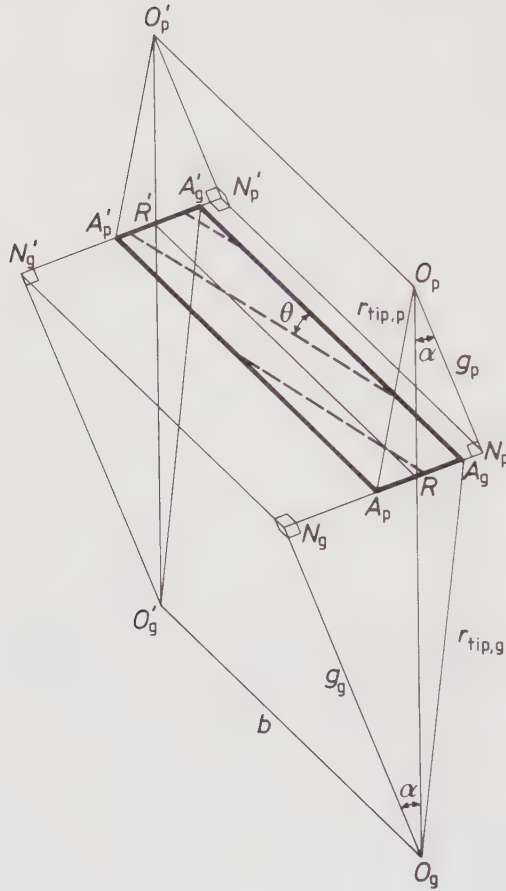


Fig. 3.7. Gear meshing region

For the tip radius this means

$$\begin{cases} r_{\text{tip},p} = a - r_{\text{bottom},g} - c_t \\ r_{\text{tip},g} = a - r_{\text{bottom},p} - c_t \end{cases} \quad (3.26)$$

where a is the centre distance and r_{bottom} is the radius of the foot circle.

From Fig. 3.2 we get

$$r_{\text{bottom}} = \frac{m_n z}{2 \cos \beta_n} - (h_{0t} - x_n) m_n \quad (3.27)$$

If the gears are mounted at a centre distance at which they will mesh tightly, the pressure angle α can be determined from the equation

$$\tan \alpha - \alpha = \tan \alpha_t - \alpha_t + 2 \frac{x_{np} + x_{ng}}{z_p + z_g} \tan \alpha_n \quad (3.28)$$

Since the base circle is unaffected by the modification process the working gear module must be

$$m = m_t \frac{\cos \alpha_t}{\cos \alpha} = \frac{m_n}{\cos \beta_n} \frac{\cos \alpha_t}{\cos \alpha} \quad (3.29)$$

The centre distance can now be written

$$a = \frac{m(z_p + z_g)}{2} = \frac{m_n}{2 \cos \beta_n} (z_p + z_g) \frac{\cos \alpha_t}{\cos \alpha} \quad (3.30)$$

Substituting Eqs. 3.25, 3.27 and 3.30 in Eq. 3.26 we obtain

$$\begin{cases} r_{tip,p} = \frac{m_n(z_p + z_g)}{2 \cos \beta_n} \frac{\cos \alpha_t}{\cos \alpha} - \frac{m_n z_g}{2 \cos \beta_n} + (1 - x_{ng}) m_n \\ r_{tip,g} = \frac{m_n(z_p + z_g)}{2 \cos \beta_n} \frac{\cos \alpha_t}{\cos \alpha} - \frac{m_n z_p}{2 \cos \beta_n} + (1 - x_{np}) m_n \end{cases} \quad (3.31)$$

Furthermore, the transverse base pitch

$$t = \pi m \cos \alpha = \pi m_t \cos \alpha_t = \frac{\pi}{\sqrt{\cos^2 \beta_n + \tan^2 \alpha_n}} \cdot m_n \quad (3.32)$$

3.4. Gear Existence Conditions

In order to get good co-operation between the pinion and the gear, certain conditions must be fulfilled.

In manufacturing, undercutting should be avoided, which means that $r_{if} \geq g$. Using Eqs. 3.5 and 3.14 we obtain, in order to avoid undercutting the tooth

$$z \geq z_{\min} = 2 \cos \beta_n \left(1 + \frac{\cos^2 \beta_n}{\tan^2 \alpha_n} \right) [h_{0t} - x_n - r_{0t} (1 - \sin \alpha_n)] \quad (3.33)$$

The number of teeth necessary to avoid undercutting the tooth, $z_{\min}$, is shown in Fig. 3.8 as a function of the helix angle β_n and the modification factor x_n .

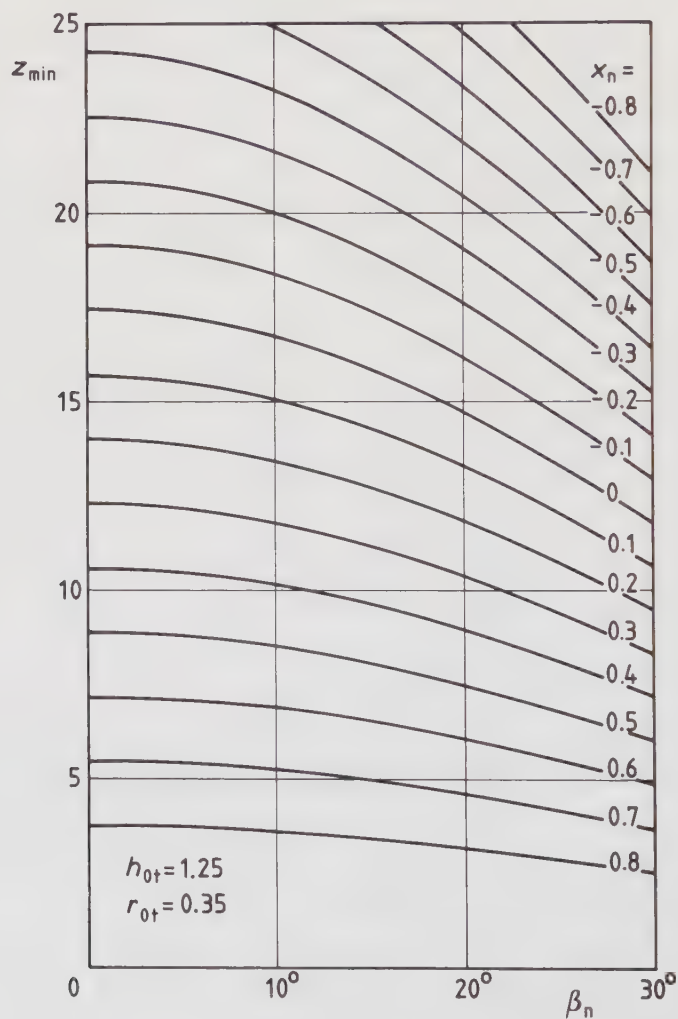


Fig. 3.8. Limit of z to avoid undercutting the tooth

Another condition that must be fulfilled in the cutting process is that the thickness of the teeth at the tip must be greater than zero. This is the case if

$$\frac{\pi + 4 x_n \tan \alpha_n}{2 z} + \tan \alpha_t - \alpha_t - \tan \left(\arccos \frac{g}{r_{\text{tip}}} \right) + \arccos \frac{g}{r_{\text{tip}}} > 0 \quad (3.34)$$

The pressure angle of the working gears, α , must be greater than zero. From Eq. 3.28 we obtain

$$\frac{x_{np} + x_{ng}}{z_p + z_g} > - \frac{\tan \alpha_t - \alpha_t}{2 \tan \alpha_n} \quad (3.35)$$

For continuous action we must have the gear contact ratio $\epsilon > 1$. From Fig. 3.7 we obtain

$$\epsilon = \frac{\sqrt{r_{\text{tip}, p}^2 - g_p^2} + \sqrt{r_{\text{tip}, g}^2 - g_g^2} - m_n \frac{z_p + z_g}{2 \cos \beta_n} \cos \alpha_t \tan \alpha}{\frac{\pi m_n}{\cos \beta_n} \cos \alpha_t} + b \frac{\sin \beta_n}{\pi m_n} \quad (3.36)$$

Interference between the pinion and the gear must be avoided in operation. If the pinion and the gear are not undercut, this requirement is met if

$$a \sin \alpha - \sqrt{r_{\text{tip}, g}^2 - g_g^2} \geq \sqrt{r_{\text{if}, p}^2 - g_p^2}$$

and

$$a \sin \alpha - \sqrt{r_{\text{tip}, p}^2 - g_p^2} \geq \sqrt{r_{\text{if}, g}^2 - g_g^2}$$

Substituting Eqs. 3.5, 3.14, 3.25, 3.30 and 3.31 in these equations, we obtain

$$\left[\frac{z_p + z_g}{2 \cos \beta_n} \frac{\cos \alpha_t}{\cos \alpha} - \frac{z}{2 \cos \beta_n} + 1 - x_n \right]^2 - \left[\frac{z_p + z_g}{2 \cos \beta_n} \cos \alpha_t - \frac{z}{2 \cos \beta_n} \cos \alpha_t \right]^2 +$$

$$- \left[\frac{z_p + z_g}{2 \cos \beta_n} \cos \alpha_t \tan \alpha - \frac{z}{2 \cos \beta_n} \sin \alpha_t + \frac{1 - x_n + c_{0t} - r_{0t} (1 - \sin \alpha_n)}{\sin \alpha_t} \right]^2 \leq 0 \quad (3.37)$$

for both the pinion and the gear.

If $c_{0t} \geq r_{0t} (1 - \sin \alpha_n)$ this condition is always fulfilled if

$$\begin{aligned}
 0 &\geq \left[\frac{z_p + z_g}{2 \cos \beta_n} \frac{\cos \alpha_t}{\cos \alpha} - \frac{z}{2 \cos \beta_n} + 1 - x_n \right]^2 - \left[\frac{z_p + z_g}{2 \cos \beta_n} \cos \alpha_t - \frac{z}{2 \cos \beta_n} \cos \alpha_t \right]^2 + \\
 &\quad - \left[\frac{z_p + z_g}{2 \cos \beta_n} \cos \alpha_t \tan \alpha - \frac{z}{2 \cos \beta_n} \sin \alpha_t + \frac{1 - x_n}{\sin \alpha_t} \right]^2 = \\
 &= -2 \frac{z}{2 \cos \beta_n} \frac{z_p + z_g}{2 \cos \beta_n} \frac{\cos \alpha_t}{\cos \alpha} [1 - \cos (\alpha - \alpha_t)] + \\
 &\quad -2 \frac{1 - x_n}{\sin \alpha_t} \frac{z_p + z_g}{2 \cos \beta_n} \frac{\cos \alpha_t}{\cos \alpha} (\sin \alpha - \sin \alpha_t) - \frac{(1 - x_n)^2}{\tan^2 \alpha_t}
 \end{aligned}$$

From this equation it is obvious that if

$$\left\{ \begin{array}{l} c_{0t} \geq r_{0t} (1 - \sin \alpha_n) \\ x_n \leq 1 \\ \alpha \geq \alpha_t \text{ i.e. } x_{np} + x_{ng} \geq 0 \end{array} \right.$$

there will be no interference between the pinion and the gear.

4. Load Distribution

In a gear reduction set consisting of two perfectly manufactured gears, mounted on exactly parallel axes, the static load distribution along and between the contact lines in the meshing plane is determined by the stiffness of the gears.

On account of local deformations of the teeth at the contact lines, these lines are transformed to contact areas, and we get a load distribution also in a plane perpendicular to the geometrical contact line, with an extension $2L$, determined by the total deformations of the gears. This distribution and its extension are assumed to be the same as with two rollers in contact.

When the teeth are submitted to the load, assuming that no part deflects except the teeth, the total deformation can be put together from bending deformation and contact deformation, where the term "bending" is used to describe the deflection of the tooth due to displacements in the tooth at a distance $>h$ below the surface.

Fig. 4.1 shows this superposition, according to the principle of Saint-Venant when the distance h is sufficiently large.

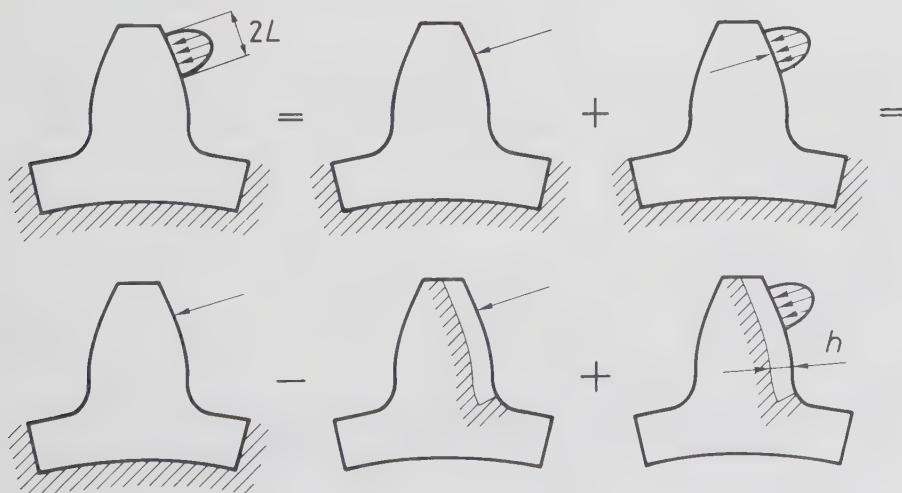


Fig. 4.1. Dividing into bending and contact conditions

4.1. Contact Deformation

By assuming Hertzian stress distribution on a straight boundary of an infinitely large plate, where the distribution along the thickness of the plate is uniform, and by integrating the strains from the surface to a distance h below the surface, Weber and Banaschek (8) derived the following formula for the displacement in the contact region

$$u_c = \frac{2(1-\nu^2)}{\pi E} \cdot q \cdot \left[\ln \left\{ \frac{h}{L} + \sqrt{1 + \left(\frac{h}{L} \right)^2} \right\} - \frac{\nu}{1-\nu} \left(\frac{h}{L} \right)^2 \left\{ \sqrt{1 + \left(\frac{L}{h} \right)^2} - 1 \right\} \right] \quad (4.1)$$

where q is the load intensity along the thickness of the plate.

Using $h \gg L$ this formula becomes

$$u_c = \frac{2(1-\nu^2)}{\pi E} \cdot q \cdot \left[\ln \frac{2h}{L} - \frac{\nu}{2(1-\nu)} \right] \quad (4.2)$$

From this equation we may determine the difference between the displacements at the depths h_1 and h_2

$$\Delta u_c = \frac{2(1-\nu^2)}{\pi E} \cdot q \cdot \ln \frac{h_1}{h_2}$$

which is the same as when the load extension $2L = 0$ (see Timoshenko and Goodier (7)). This means that the displacement at a distance $h \gg L$ from the acting force is independent of the distribution of this force, which was used in Fig. 4.1.

Substituting

$$L = \sqrt{\frac{4}{\pi} (k_p + k_g) \cdot q \cdot \rho} \quad (4.3)$$

where the equivalent radius of curvature

$$\rho = \frac{\rho_p \cdot \rho_g}{\rho_p + \rho_g} \quad (4.4)$$

and the material constant

$$k_i = \frac{1 - \nu_i^2}{E_i} \quad i = p, g \quad (4.5)$$

in Eq. 4.2, using $h = h_p = h_g$, we obtain the total displacement due to the Hertzian stress distribution

$$u_c = \frac{k_p + k_g}{\pi} \cdot q \cdot \left[\ln \frac{\pi h^2}{(k_p + k_g) \rho q} - \frac{k_p \cdot \frac{\nu_p}{1 - \nu_p} + k_g \cdot \frac{\nu_g}{1 - \nu_g}}{k_p + k_g} \right] \quad (4.6)$$

Eq. 4.1 is a solution to a two-dimensional problem, i.e. the load is assumed only to give contributions to the displacement in the same position along the contact line as that where the displacement is measured. This assumption is discussed in appendix A1.

4.2. Bending Deformation

The displacements due to the bending deformation have been calculated on an electronic computer UNIVAC 1100, using the finite element method program EUFEMI.

Fig. 4.2 shows the element divisions in the two first cases in the lower row in Fig. 4.1, from which the bending displacements at the tooth surface are obtained by subtracting the contact zone displacements from the tooth displacements for a number of force positions, determined by the current number of nodes at the involute flank in Fig. 4.2, at the surface of the tooth.

Since these numerical values of the displacements are evaluated in a limited number of points we have to interpolate the displacement to the current load position when the load distribution is calculated. For this reason an approximate equation has been built up using the numerical results. This equation must satisfy the reciprocal theory and with this in mind the equation is built up as

$$\alpha(\xi_F, \eta_F, \xi, \eta) = \sqrt{\psi(\xi_F, \eta_F) \psi(\xi, \eta)} \Gamma(|\xi - \xi_F|, \eta \cdot \eta_F, \eta + \eta_F, \eta^2 + \eta_F^2) \quad (4.7)$$

describing the displacement in a point with the coordinates ζ and η on account of a unit force in a point with the coordinates ζ_F and η_F . The function $\Gamma = 1$ when $\zeta = \zeta_F$, making the function ψ describe the displacement in the same point as that where the unit force acts.

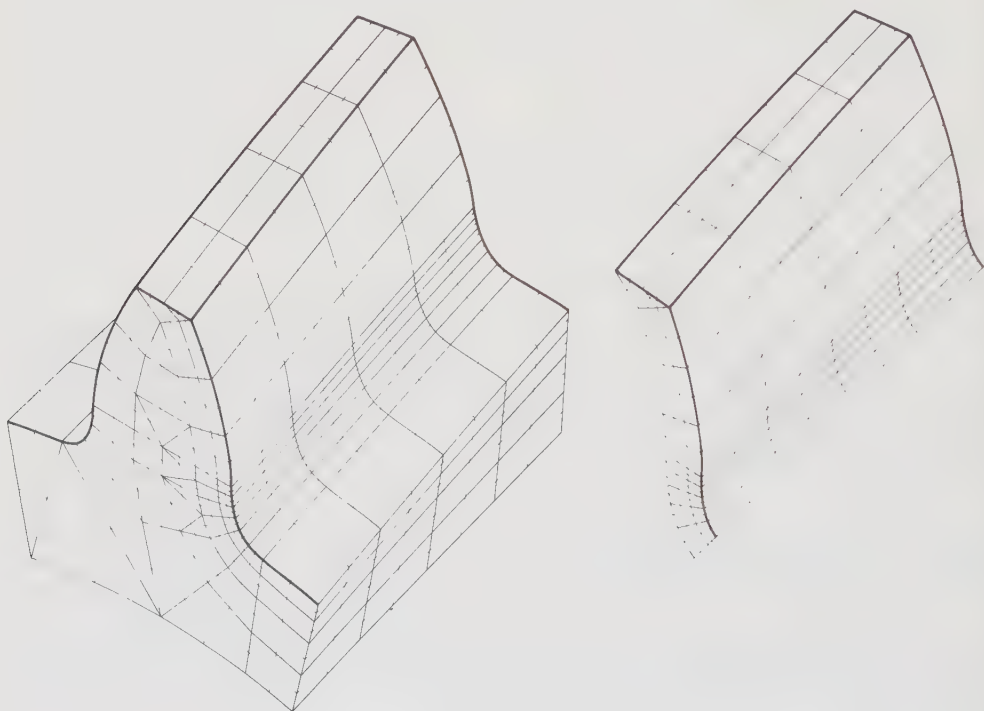


Fig. 4.2. Helical gear tooth and contact zone divided into finite elements

The function ψ is put together from two functions, each depending on the distance from one of the free sides of the gear and since the determination of these functions has been carried out for a face width where these functions are independent of each other, the function ψ can be used for every face width.

An example of the function ψ and the numerical values received from the finite element program, used to determine ψ in this example, is shown in Fig. 4.3.

For the function Γ , in principle, a decreasing exponential function is used where, see Eq. 4.7, the exponent is dependent on the height position at the tooth. In Fig. 4.4 an example of the function Γ is shown.

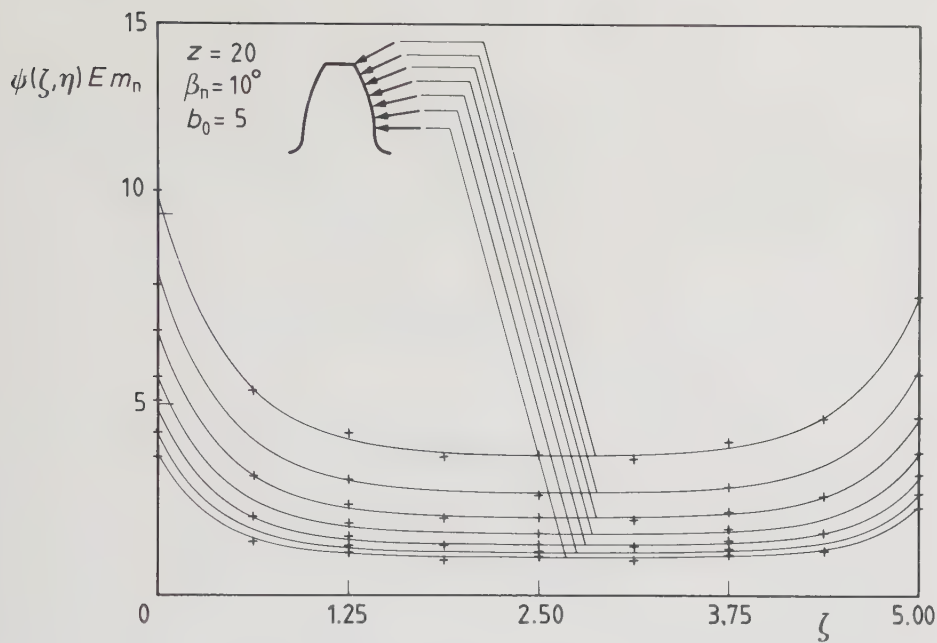


Fig. 4.3. Approximate function $\psi(\zeta, \eta)$ and corresponding nodal points displacement

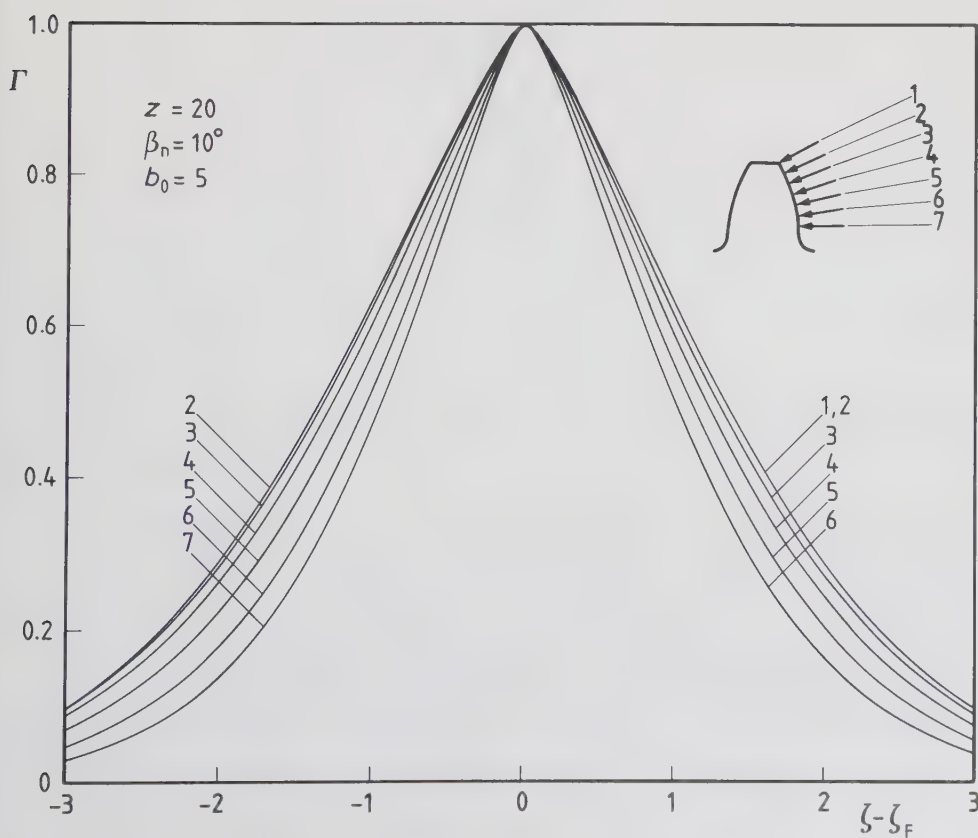


Fig. 4.4. Approximate function Γ

The displacements have been calculated for the gears with $z = z_{min}$, 20, 25, 30, 40, 80 and 160 when $\beta_n = 0, 10^\circ$ and 20° respectively, $x_{np} = x_{ng} = 0$ and $b_0 = 5$. The equations for ψ and Γ for these gears, used in the further calculations of the load distribution, are given in appendix A2.

4.3. Condition of Deformation

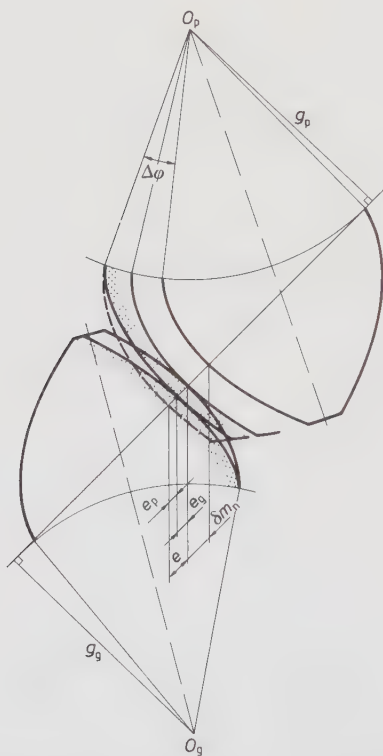


Fig. 4.5. Deformation of a pair of teeth in contact

Fig. 4.5 shows a pair of teeth in a position, in the transverse plane, where the gears are not loaded. If, from this position, the pinion is twisted an angle $\Delta\varphi$ when the gear is fixed, the total displacement of the teeth is $e = e_p + e_g = g_p \Delta\varphi - \delta m_n$, where e_p and e_g are the displacements of the teeth of the pinion and the gear respectively and δ the non-dimensional contact error. This is valid at a certain contact point in a certain transverse plane, but if all simultaneous points of contact are to be in contact even after deformation due to the angle of

twist $\Delta\varphi = \text{constant}$, we obtain $e + \delta m_n = \text{constant}$.

By using the displacements perpendicular to the contact lines, u , the condition of deformation becomes

$$\frac{u}{\cos \theta} + \delta m_n = \Delta m_n \quad (4.8)$$

where Δ is a non-dimensional constant.

4.4. Load Distribution Solution

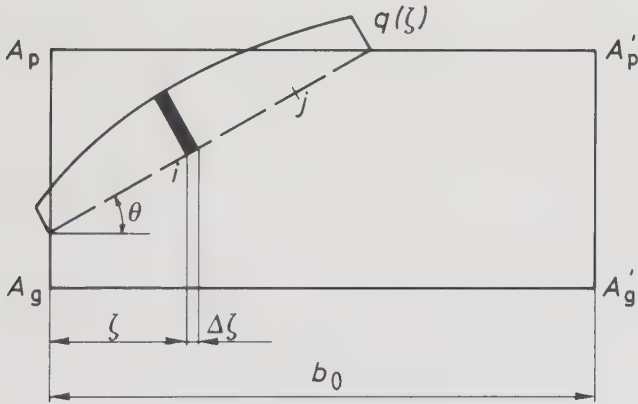


Fig. 4.6. Force acting on a contact line

Fig. 4.6 shows a meshing plane with one loaded contact line. The displacement of the teeth in contact due to the contact force $F_i = q(\xi) \cdot \Delta\xi \cdot m_n / \cos \theta$ is in the point i $\alpha_{ii}F_i + u_{ci}$ and in the point j $\alpha_{ij}F_i$.

If i and j are two of n discrete points on the contact line, the total displacement in the point i can be written

$$u_i = \sum_{j=1}^n (\alpha_{pij} + \alpha_{gij}) F_j + u_{ci} = \sum_{j=1}^n \alpha_{ij} F_j + u_{ci} \quad (4.9)$$

The total normal force in the meshing plane is

$$F = \sum_{k=1}^K \int_0^{b_0} q^{(k)}(\xi) \frac{m_n}{\cos \theta} d\xi = \sum_{k=1}^K Q^{(k)} = \sum_{k=1}^K \sum_{j=1}^{n_k} F_j^{(k)} \quad (4.10)$$

where K is the number of simultaneous contact lines in the current position and $q \equiv 0$ outside the boundaries of the meshing plane.

By setting

$$\left\{ \begin{array}{l} u = u_o \frac{F}{E_p m_n} \\ \alpha_{ij} = \frac{\alpha_{oij}}{E_p m_n} = \frac{\alpha_{opij}}{E_p m_n} + \frac{\alpha_{ogij}}{E_g m_n} \\ q = q_o \frac{F}{m_n} \\ F = F_o E_p m_n^2 \end{array} \right. \quad (4.11)$$

and using Eq. 4.6 with

$$\left\{ \begin{array}{l} \rho = \rho_o m_n \\ h = h_o m_n \end{array} \right. \quad (4.12)$$

we obtain from Eq. 4.9

$$u_{oi} = \sum_{j=1}^n \alpha_{oij} \frac{F_j}{F} + q_{oi} \frac{k_p + k_g}{\pi} E_p \cdot \\ \cdot \left[\ln \left(\frac{\pi h_o^2}{(k_p + k_g) E_p} \frac{1}{\rho_{oi} q_{oi} F_o} \right) - \frac{\frac{\nu_p}{1 - \nu_p} + \frac{k_g}{k_p} \frac{\nu_g}{1 - \nu_g}}{1 + \frac{k_g}{k_p}} \right] \quad (4.13)$$

If, in the interval $[i, i + 1]$, q is approximated by the mean value of one parabola through the points i , $i + 1$ and $i + 2$, and another parabola through the points $i - 1$, i and $i + 1$, we obtain

$$F_j = \sum_{l=1}^n \phi_{jl} q_l \quad (4.14)$$

where $\phi_{jl} \equiv 0$ when $|l-j| > 2$.

Setting $\phi = \phi_o m_n$ we obtain from Eq. 4.13

$$\begin{aligned}
 u_{oi} &= \sum_{j=1}^n (\alpha_{oj} \sum_{l=1}^n \phi_{olj} q_{ol}) + u_{oci} = \sum_{j=1}^n \sum_{l=1}^n \alpha_{oj} \phi_{olj} q_{ol} + u_{oci} = \\
 &= \sum_{j=1}^n \sum_{l=1}^n \alpha_{oil} \phi_{olj} q_{oj} + u_{oci} = \\
 &= \sum_{j=1}^n (q_{oj} \sum_{l=j-2}^{j+2} \alpha_{oil} \phi_{olj}) + q_{oi} \frac{1 - \nu_p^2 + \frac{E_p}{E_g} (1 - \nu_g^2)}{\pi} \cdot \\
 &\cdot \left[\ln \left(\frac{\pi h_o^2}{1 - \nu_p^2 + \frac{E_p}{E_g} (1 - \nu_g^2)} \frac{1}{\rho_{oi} q_{oi} F_o} \right) - \frac{\frac{\nu_p}{1 - \nu_p} + \frac{k_g}{k_p} \frac{\nu_g}{1 - \nu_g}}{1 + \frac{k_g}{k_p}} \right]
 \end{aligned}$$

and from Eq. 4.10

$$\begin{aligned}
 \sum_{k=1}^K \sum_{j=1}^{n_k} \frac{F_j^{(k)}}{F} &= \sum_{k=1}^K \sum_{j=1}^{n_k} \sum_{l=1}^{n_k} \phi_{olj}^{(k)} q_{ol}^{(k)} = \sum_{k=1}^K \sum_{j=1}^{n_k} \sum_{l=1}^{n_k} \phi_{olj}^{(k)} q_{oj}^{(k)} = \\
 &= \sum_{k=1}^K \sum_{j=1}^{n_k} (q_{oj}^{(k)} \sum_{l=j-2}^{j+2} \phi_{olj}^{(k)}) = 1
 \end{aligned}$$

In order to simplify the equations we set

$$\left\{ \begin{aligned} \psi_{ij}^{(k)} &= \sum_{l=j-2}^{j+2} \alpha_{oil}^{(k)} \phi_{olj}^{(k)} \\ \Gamma_j^{(k)} &= \sum_{l=j-2}^{j+2} \phi_{olj}^{(k)} \\ A &= \frac{1 - \nu_p^2 + \frac{E_p}{E_g} (1 - \nu_g^2)}{\pi} \\ B &= A \left[\ln \frac{h_o^2}{A} - \frac{\frac{\nu_p}{1 - \nu_p} + \frac{k_g}{k_p} \frac{\nu_g}{1 - \nu_g}}{1 + \frac{k_g}{k_p}} \right] \end{aligned} \right.$$

which with Eq. 4.8 give

$$\begin{cases} \sum_{j=1}^{n_k} q_{oj}^{(k)} \psi_{ij}^{(k)} + q_{oi}^{(k)} [B - A \ln (\rho_{oi}^{(k)} q_{oi}^{(k)} F_o)] = \frac{\Delta - \delta_i^{(k)}}{F_o} \cos \theta \\ \sum_{k=1}^K \sum_{j=1}^{n_k} q_{oj}^{(k)} \Gamma_j^{(k)} = 1 \end{cases} \quad (4.15)$$

The first of these equations must be satisfied in all points of simultaneous contact and together with the last equation the number of equations is equal to the number of unknowns. One method to solve this system of equations is to divide the first equation in Eq. 4.15 by the constant Δ , from which we obtain

$$\sum_{j=1}^{n_k} \frac{q_{oj}^{(k)}}{\Delta} \psi_{ij}^{(k)} + \frac{q_{oi}^{(k)}}{\Delta} [B - A \ln (\rho_{oi}^{(k)} q_{oi}^{(k)} F_o)] = \frac{1 - \frac{\delta_i^{(k)}}{\Delta}}{F_o} \cos \theta \quad (4.16)$$

Setting out from a suitable assumption of $q_{oj}^{(k)}$ and Δ , for a given value of F_o , $q_{oj}^{(k)}/\Delta$ can be determined in all points from Eq. 4.16. The last equation in Eq. 4.15 then gives

$$\Delta = \frac{1}{\sum_{k=1}^K \sum_{j=1}^{n_k} \frac{q_{oj}^{(k)}}{\Delta} \Gamma_j^{(k)}}$$

from which a new solution of $q_{oj}^{(k)}$ is obtained. By repetition $q_{oj}^{(k)}$ can be determined with the required accuracy.

In the above it is assumed that the teeth will not separate in any contact point, but to deal with this we set $q_{oj}^{(k)} = 0$ if $\delta_i^{(k)} \geq \Delta$. Numerically this can be done by setting $\delta_i^{(k)} = \Delta$ and multiplying $q_{oi}^{(k)}$ in the same equation by a large number, which leaves the rest of the equations unaffected, and force $q_{oi}^{(k)}$ to be very small, nearly zero.

In Figs. 4.7 to 4.22 the load intensities in the meshing planes are shown graphically for some examples of gear sets, calculated in the way previously described.

In all numerical calculations $E_p = E_g$ and $\nu_p = \nu_g = 0.3$ are used.

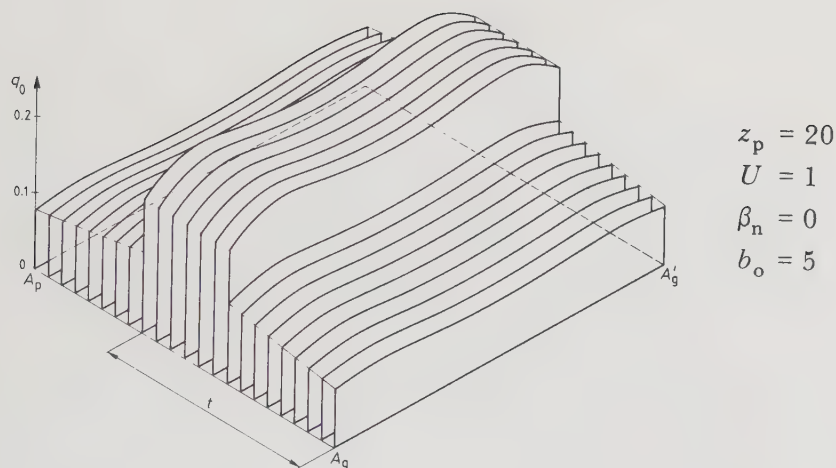


Fig. 4.7. Load distribution when $\delta_1 = 0$ and $F_o = 1.12 \cdot 10^{-3}$

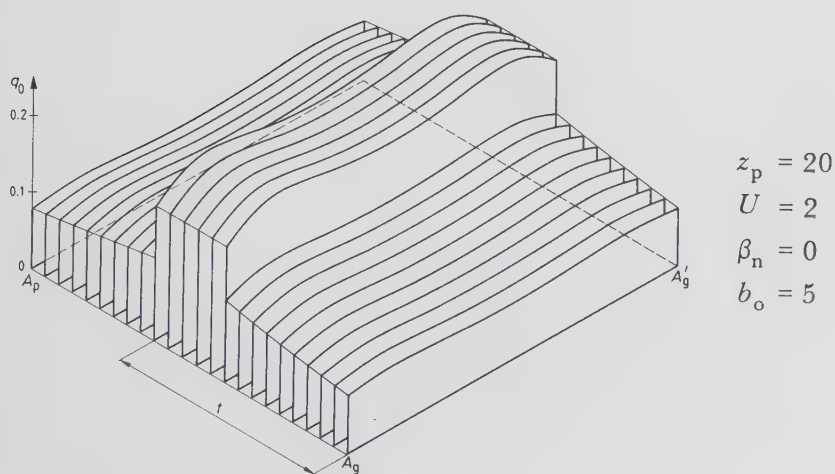


Fig. 4.8. Load distribution when $\delta_1 = 0$ and $F_o = 1.37 \cdot 10^{-3}$

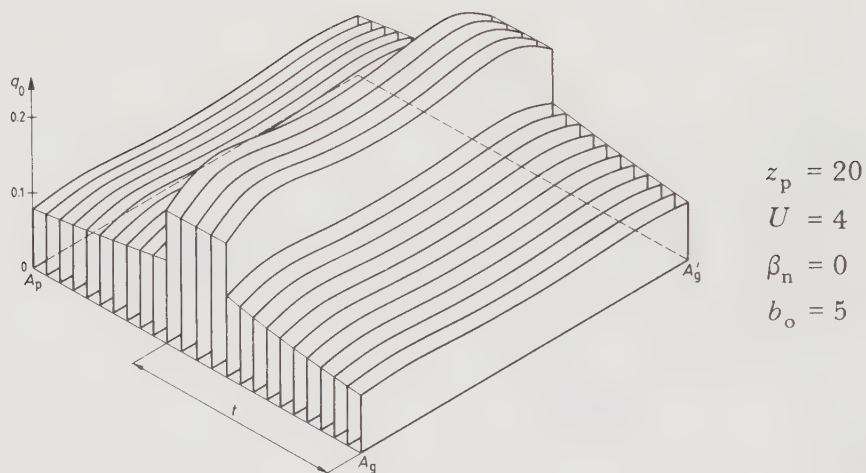


Fig. 4.9. Load distribution when $\delta_1 = 0$ and $F_o = 1.23 \cdot 10^{-3}$

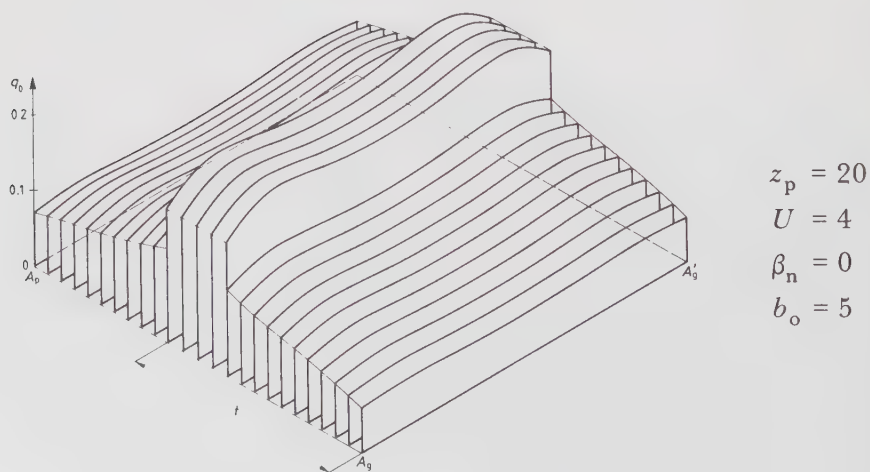


Fig. 4.10. Load distribution when $\delta_1 = 0.001$ and $F_o = 1.58 \cdot 10^{-3}$

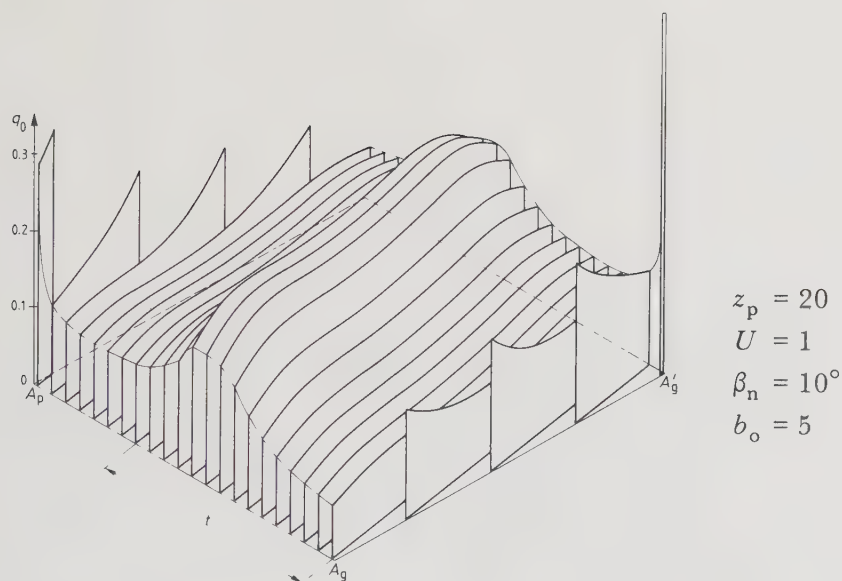


Fig. 4.11. Load distribution when $\delta_1 = 0$ and $F_o = 3.09 \cdot 10^{-4}$

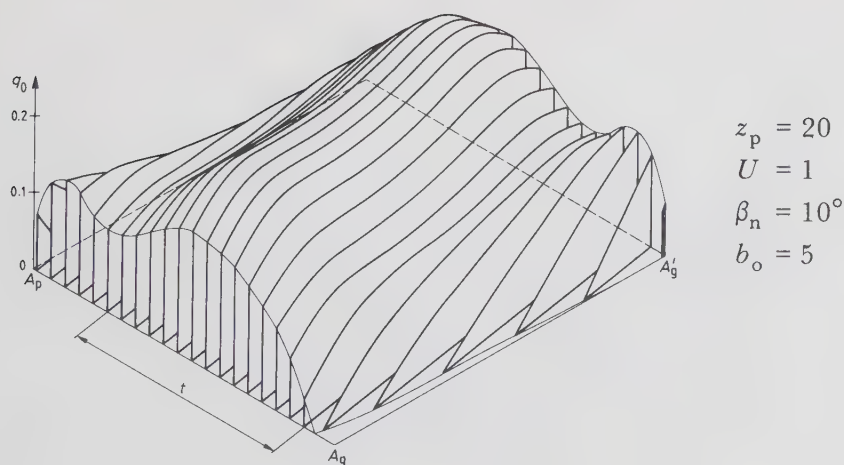


Fig. 4.12. Load distribution when $\delta_1 = 0.007$ and $F_o = 1.21 \cdot 10^{-3}$

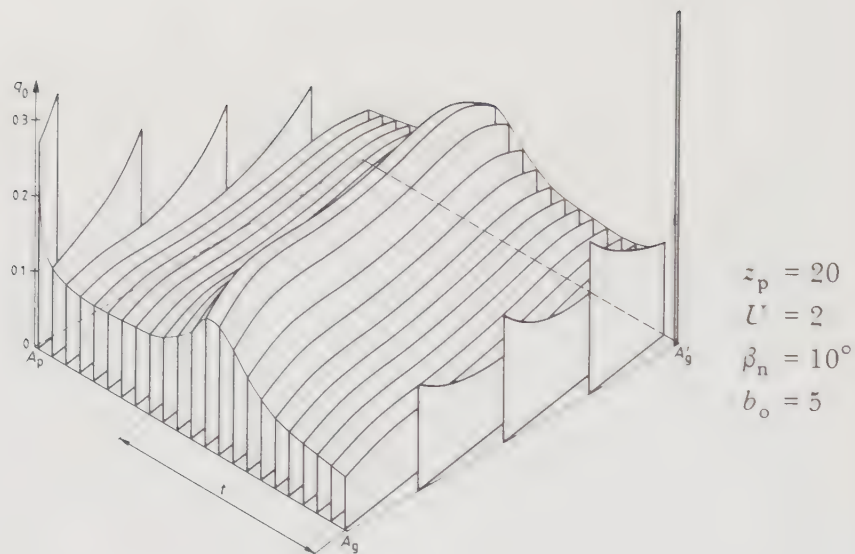


Fig. 4.13. Load distribution when $\delta_1 = 0$ and $F_o = 3.03 \cdot 10^{-4}$

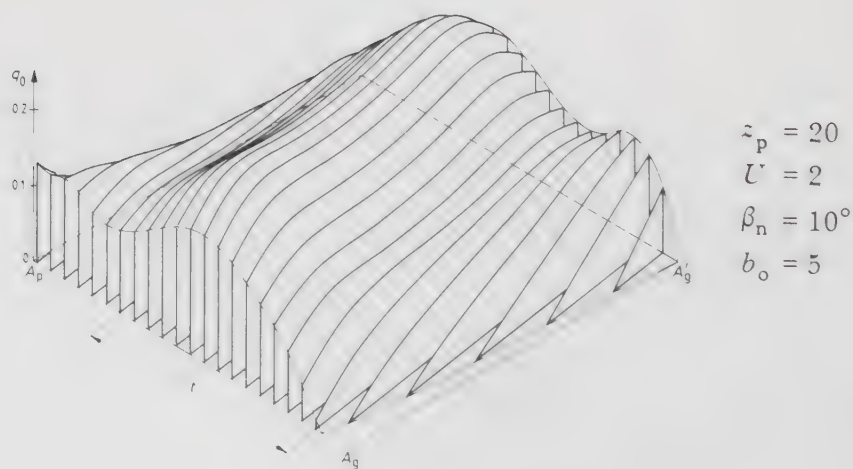


Fig. 4.14. Load distribution when $\delta_1 = 0.006$ and $F_o = 1.50 \cdot 10^{-3}$

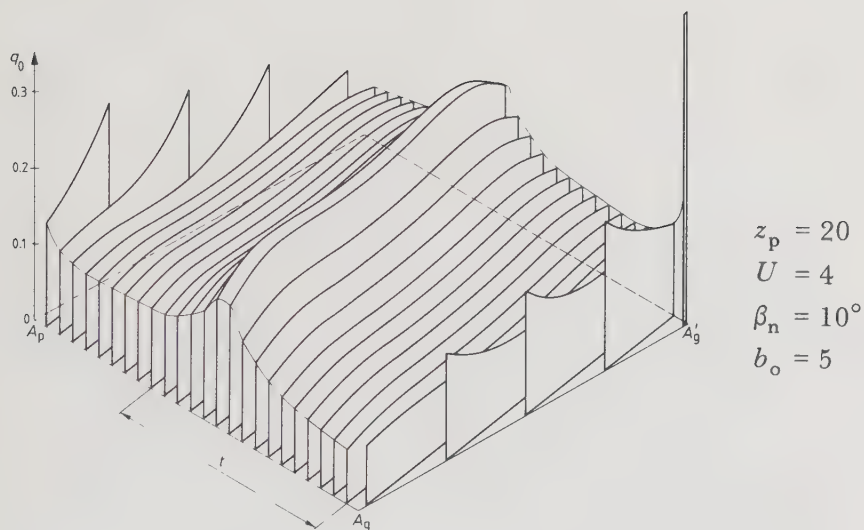


Fig. 4.15. Load distribution when $\delta_1 = 0$ and $F_o = 2.84 \cdot 10^{-4}$

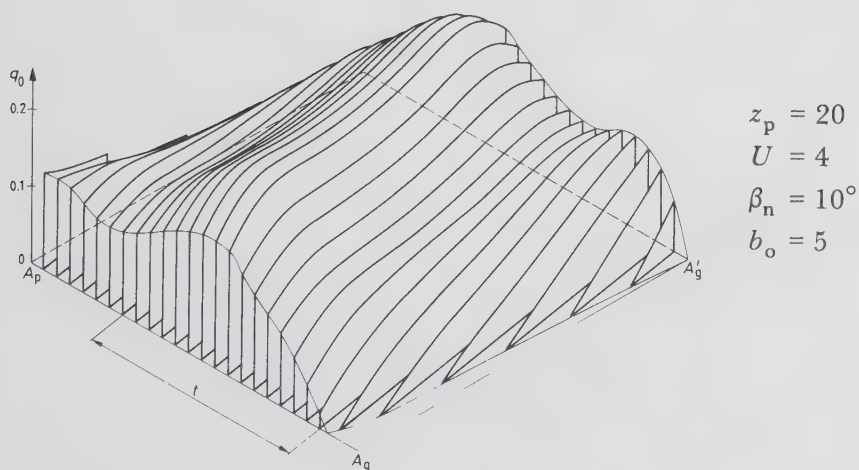


Fig. 4.16. Load distribution when $\delta_1 = 0.005$ and $F_o = 1.66 \cdot 10^{-3}$

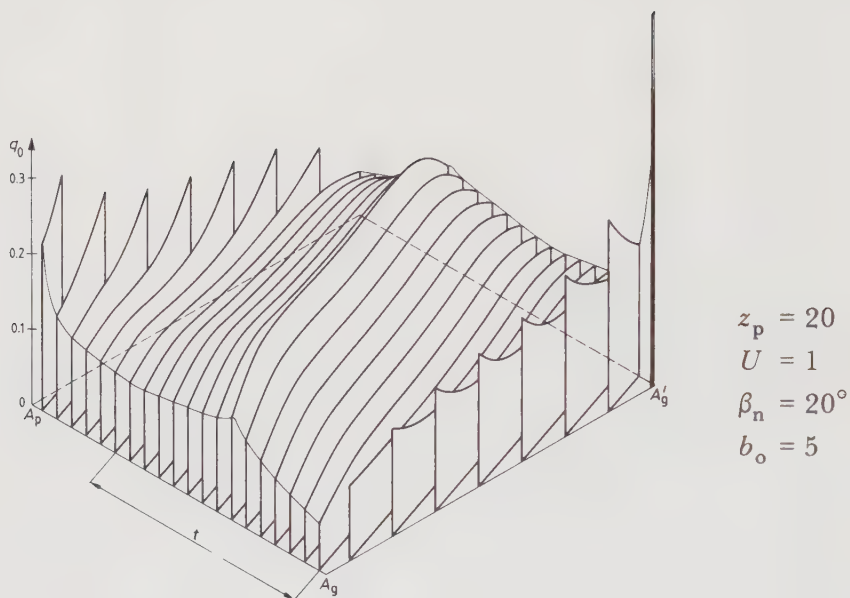


Fig. 4.17. Load distribution when $\delta_1 = 0$ and $F_o = 3.89 \cdot 10^{-4}$

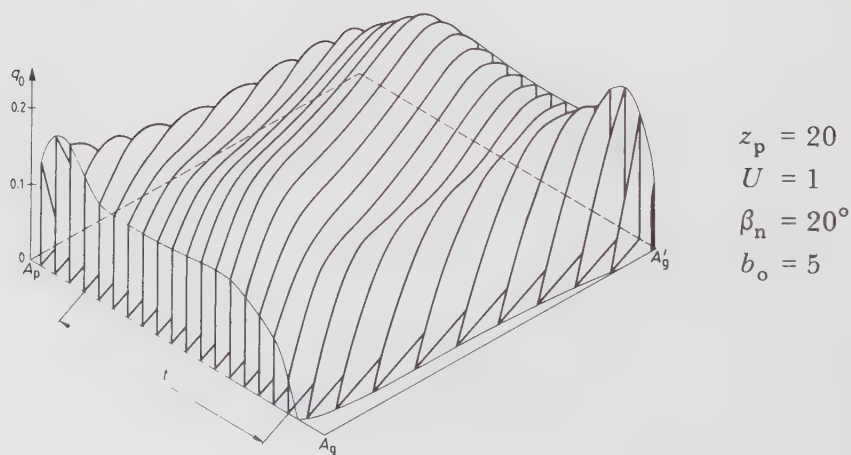


Fig. 4.18. Load distribution when $\delta_1 = 0.006$ and $F_o = 1.21 \cdot 10^{-3}$

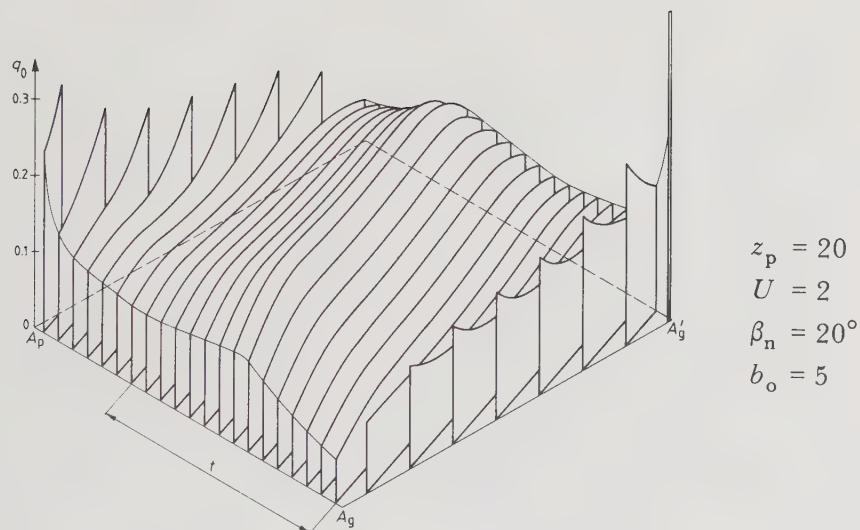


Fig. 4.19. Load distribution when $\delta_1 = 0$ and $F_o = 4.52 \cdot 10^{-4}$

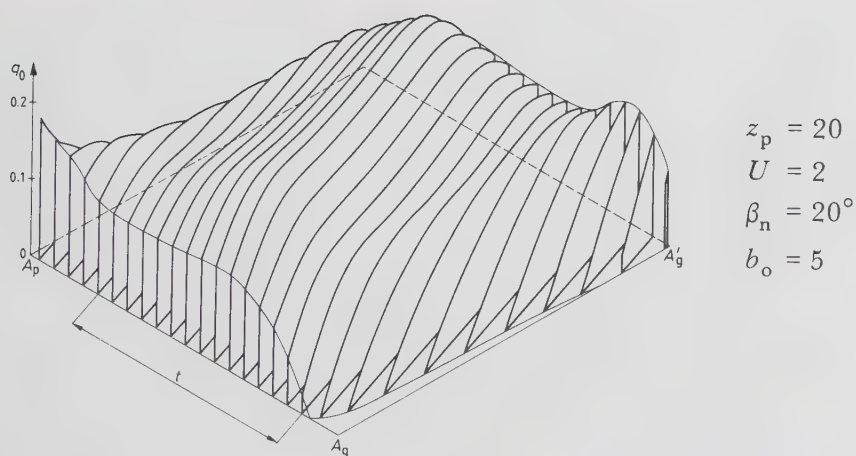


Fig. 4.20. Load distribution when $\delta_1 = 0.004$ and $F_o = 1.51 \cdot 10^{-3}$

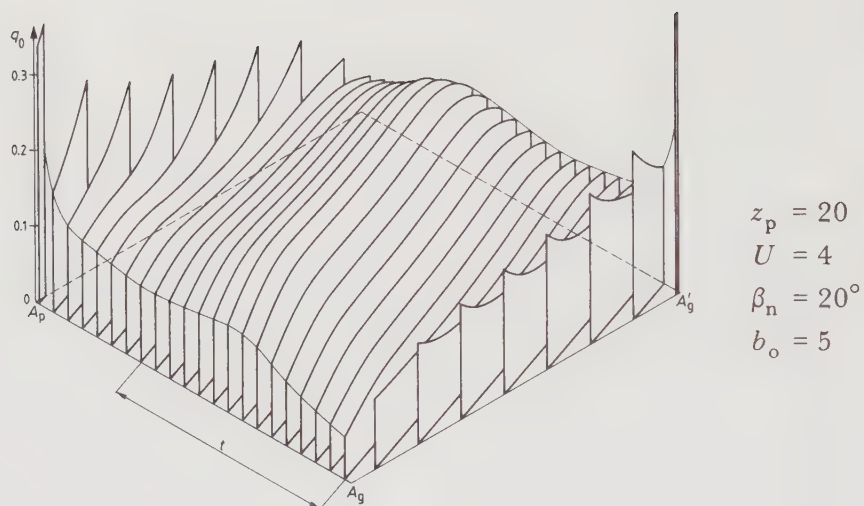


Fig. 4.21. Load distribution when $\delta_1 = 0$ and $F_o = 4.80 \cdot 10^{-4}$

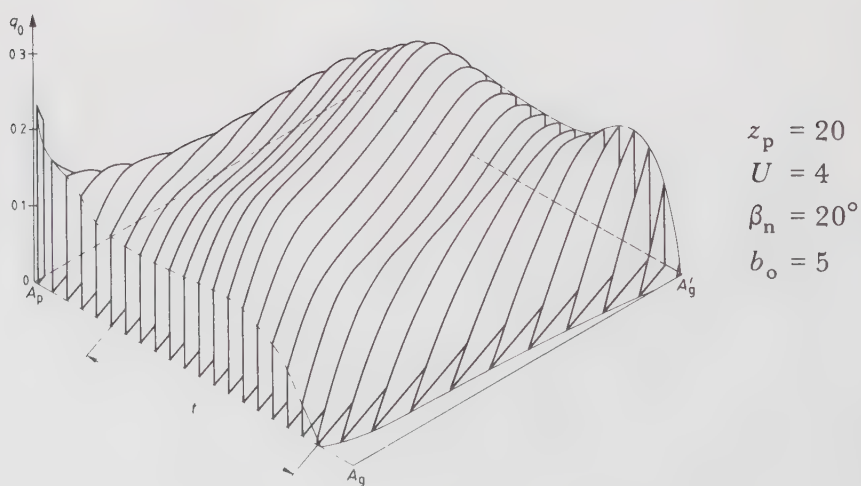


Fig. 4.22. Load distribution when $\delta_1 = 0.004$ and $F_o = 1.64 \cdot 10^{-3}$

In order to check the accuracy of the previously described way of calculating the distribution of the load, the spur gear load distribution can be calculated using a two-dimensional finite element division which can be solved at a reasonable cost with a considerably finer element division than the three-dimensional one. The equations needed to calculate the load distribution in this special case, where the load intensity in the ξ -direction is constant, can be derived from the previous equations by setting

$$\left\{ \begin{array}{l} n_k = 1 \\ \phi = b \\ \psi^{(k)} = b_o \cdot \alpha_o^{(k)} \\ \Gamma = b_o \end{array} \right.$$

which, inserted in Eq. 4.15, give

$$\left\{ \begin{array}{l} \frac{q_o^{(k)}}{\Delta} = \frac{1 - \frac{\delta^{(k)}}{\Delta}}{F_o [b_o \alpha_o^{(k)} + B - A \ln (\rho_o^{(k)} q_o^{(k)} F_o)]} \\ \Delta = \frac{1}{b_o \sum_{k=1}^K \frac{q_o^{(k)}}{\Delta}} \end{array} \right. \quad (4.17)$$

One case has been solved, namely the gear set with $z_p = z_g = 20$, and when comparing the distributed load in different positions in the meshing plane with the factor Q/F , see Eq. 4.10, for the corresponding three-dimensional solution, graphically represented in Fig. 4.7, it is found that the deviation is less than 2 per mille.

5. Gear Contact Pressures

Now the problem is to design an external involute helical gear set with the gear ratio $U = z_g/z_p$, the centre distance a and the face width b to carry the load from the torque M_p at the pinion shaft without breakage of the gear flanks.

Hertz's theories give the maximum contact pressure p between two cylinders loaded to each other with the load intensity q

$$p = \sqrt{\frac{q}{\pi (k_p + k_g) \rho}} \quad (5.1)$$

The torque M_p , in terms of the total normal force F and the geometry, becomes

$$M_p = F \cdot \cos \theta \frac{m_n z_p}{2 \cos \beta_n} \cos \alpha_t = F \frac{m_n z_p}{2} \cos \alpha_n \quad (5.2)$$

which, together with Eq. 3.30, inserted in Eq. 5.1 gives

$$\begin{aligned} p &= \sqrt{\frac{q_o \frac{F}{m_n}}{\pi (k_p + k_g) \rho_o m_n}} = \\ &= \sqrt{\frac{q_o}{\pi (k_p + k_g) \rho_o} \frac{2M_p}{z_p \cos \alpha_n} \frac{b_o}{b} \frac{z_p^2 (1+U)^2 \cos^2 \alpha_t}{4a^2 \cos^2 \beta_n \cos^2 \alpha}} \end{aligned}$$

The non-dimensional contact pressure now becomes

$$p_o = p \sqrt{\frac{a^2 b (k_p + k_g)}{M_p}} = \frac{1+U}{\cos \beta_n} \frac{\cos \alpha_t}{\cos \alpha} \sqrt{\frac{z_p \cdot b_o \cdot q_o}{2\pi \rho_o \cos \alpha_n}} \quad (5.3)$$

from which we obtain the maximum non-dimensional contact pressure

$$p_{o, \max} = \frac{1+U}{\cos \beta_n} \frac{\cos \alpha_t}{\cos \alpha} \sqrt{\frac{z_p \cdot b_o}{2\pi \cos \alpha_n} \left(\frac{q_o}{\rho_o} \right)_{\max}} \quad (5.4)$$

If the value of F_o is needed in the determination of the load distribution q , F_o must be of a magnitude to give the permissible contact pressure $p_{\max}$ of the gear material. To obtain this value of F_o we use the definition of p_o in Eq. 5.3

$$p_{o, \max}^2 = p_{\max}^2 \frac{m_n^2 z_p^2 (1+U)^2 \cos^2 \alpha_t}{4 \cdot \cos^2 \beta_n \cdot \cos^2 \alpha} b_o m_n \frac{\frac{1-\nu_p^2}{E_p} + \frac{1-\nu_g^2}{E_g}}{F \frac{m_n z_p}{2} \cos \alpha_n}$$

from which

$$F_o = \frac{F}{E_p m_n^2} = \left(\frac{p_{\max}}{E_p} \right)^2 \frac{1-\nu_p^2 + \frac{E_p}{E_g} (1-\nu_g^2)}{2 \cdot p_{o, \max}^2} \frac{z_p \cdot b_o (1+U)^2 \cos^2 \alpha_t}{\cos^2 \beta_n \cos \alpha_n \cos^2 \alpha} \quad (5.5)$$

In the calculations Eq. 5.5 is used so that when $(q_o/\rho_o)_{\max}$ has been determined for some value of F_o this value is corrected and a new value of $(q_o/\rho_o)_{\max}$ is obtained and so on, until F_o from Eq. 5.5 gives the predicted value. In the calculations $p_{\max}/E_p = 0.005$ is used, which corresponds to $p_{\max} \simeq 1000 \text{ MN/m}^2$ for gears of steel.

The equivalent radius of curvature ρ_o has its minimum value at the boundary $A_g A_g'$ of the meshing plane.

In the spur gear case the load intensity at this boundary is about 40 per cent of the maximum value when one tooth carries the whole load, i.e. $q_o \neq q_{o, \max}$ when $\rho_o = \rho_{o, \min}$ and vice versa. For small values of z_p $p_{o, \max}$ occurs at the boundary $A_g A_g'$ for some values of U , but with gears with tip relief the load intensity here is decreased, so $p_{o, \max}$ is moved to the line where one pair of teeth are in contact and is located nearest $A_g A_g'$, i.e. one transverse base pitch from the boundary $A_p A_p'$. The load intensity here is unaffected by the tip

relief since $\epsilon < 2$ for all spur gears studied.

When b_o is increased $p_{o, \max}$ increases but this difference is small. A comparison shows that when b_o is changed from 5 to 15 $p_{o, \max}$ increases less than 1.3 per cent, for which reason the variation in b_o can be disregarded when $\beta_n = 0$.

In Fig. 5.1 $p_{o, \max}$ for spur gears is presented as a function of U where the values of $p_{o, \max}$ have been calculated for $b_o = 15$.

When $\beta_n > 0$ the problem is somewhat more difficult. In the previous section it was found that, when the gear flanks were unrelieved, the maximum value of q was in the point A_g' . However it is not possible to determine q_o in this point since the contact line here has no length. The calculations are therefore performed for a shortest length of the contact line near the point A_g' , here set to $0.05 m_n / \cos \theta$.

When the gears have been running a while it is reasonable to believe that the increase of q_o at the boundaries of the meshing plane has disappeared. Therefore it could be interesting to compare the shortest line contact pressure with the contact pressure from the somewhat smoother uniform load distribution. In (9) Wennerström has calculated this contact pressure, but in order to be able to disregard the influence of ϵ he made the calculation with $b = a$.

In the uniform load distribution case, q_o is greatest when the total length of the simultaneous contact lines, L_{tot} , has its minimum value, which happens when one contact line just touches the point A_g' or A_p , among other positions. The greatest value of p_o , $p_{o, \max}$, occurs in this position in all contact points at the boundary $A_g A_g'$, for instance in point A_g' . $p_{o, \max}$ is obtained from Eq. 5.4 by setting $q_o = m_n / L_{tot}$.

To avoid the maximum contact pressure at the boundary $A_g A_g'$, where the radius of curvature has its lowest value, tip relief can be taken into account. The contact pressure for relieved gears has been calculated for the lowest value of $\delta_1 = 0, 0.001, 0.002, \dots$ and $\delta_2 = 0.5$ where $p_{o, \max}$ does not occur at the boundaries $A_g A_g'$ and $A_p A_p'$.

The contact pressures for both unrelieved and relieved gears are tabled in appendix A3 where also the size of the tip relief is given.

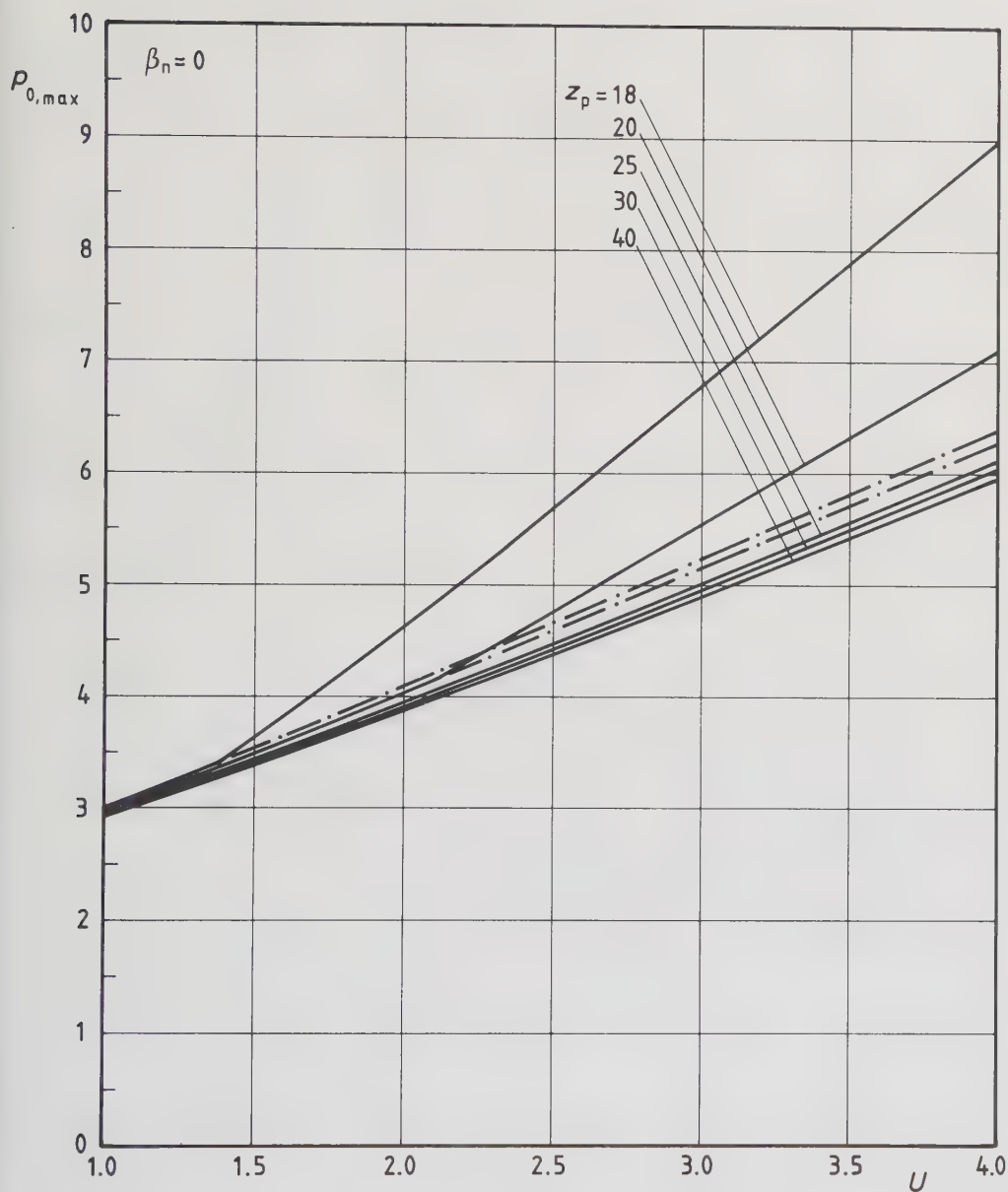


Fig. 5.1. Spur gear contact pressures for gears without tip relief (unbroken lines) and with tip relief (dot-and-dashed lines)

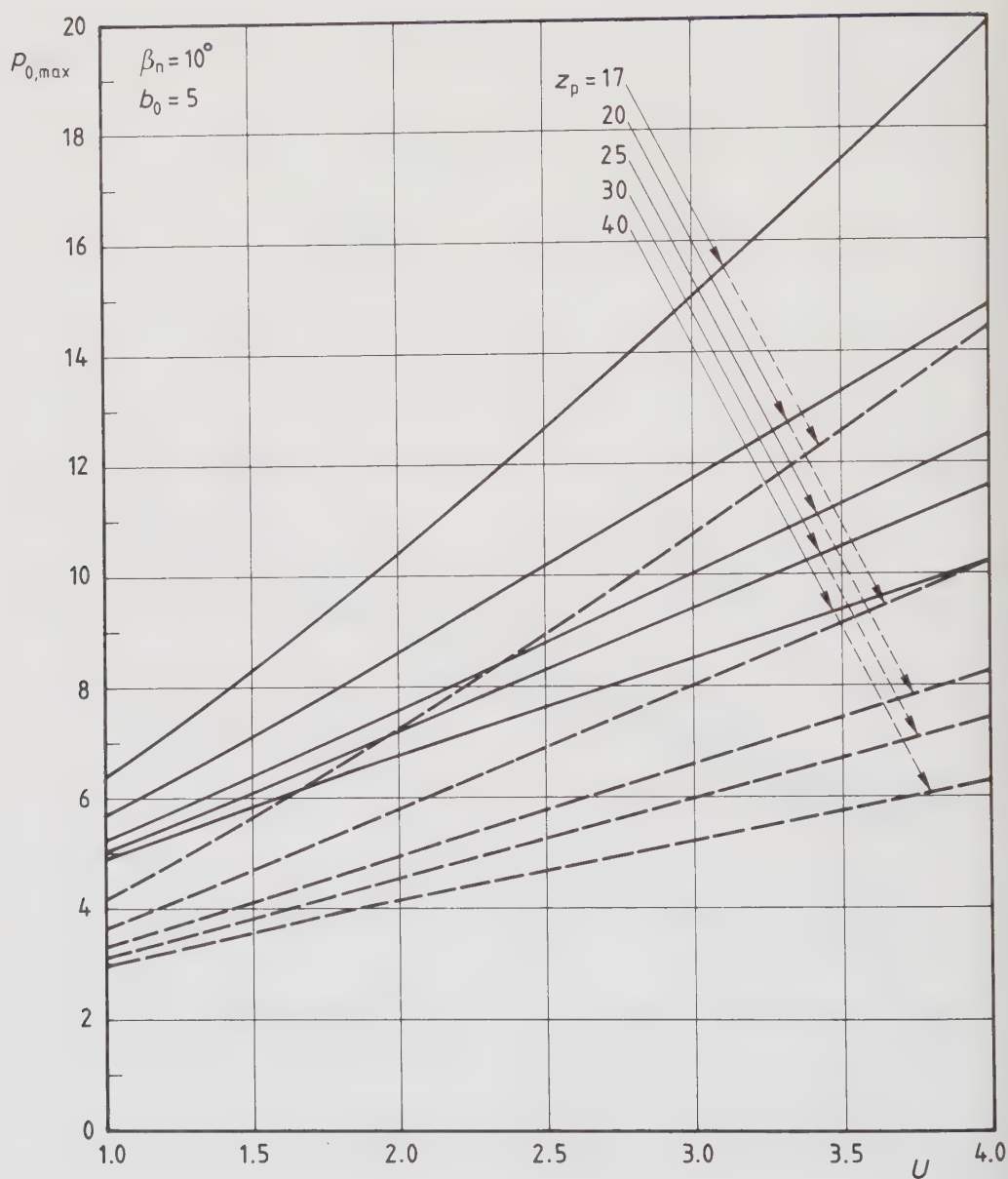


Fig. 5.2. Helical gear contact pressures for gears without tip relief (unbroken lines), compared with contact pressures from uniform load distributions (dashed lines)

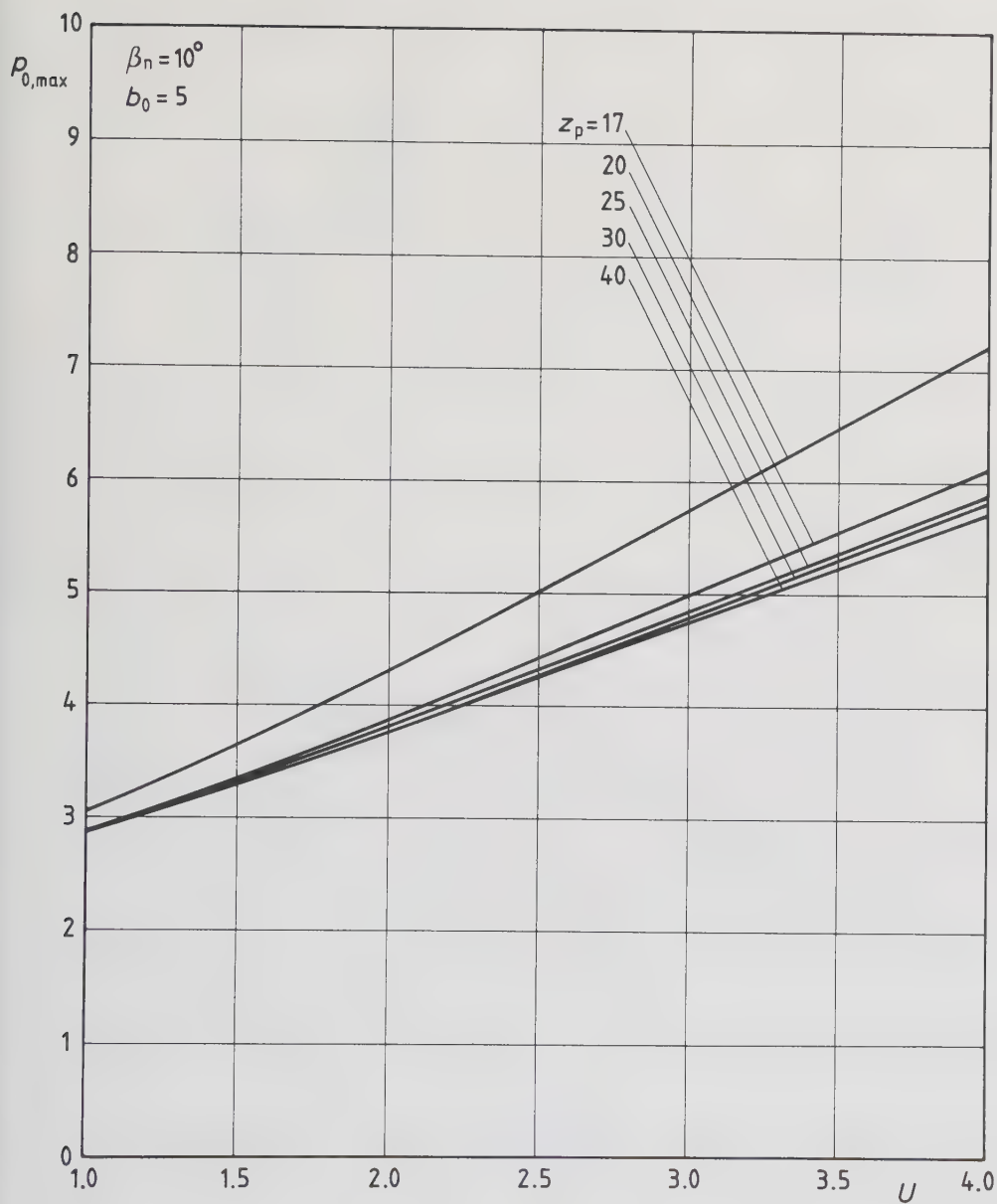


Fig. 5.3. Helical gear contact pressures for gears with tip relief

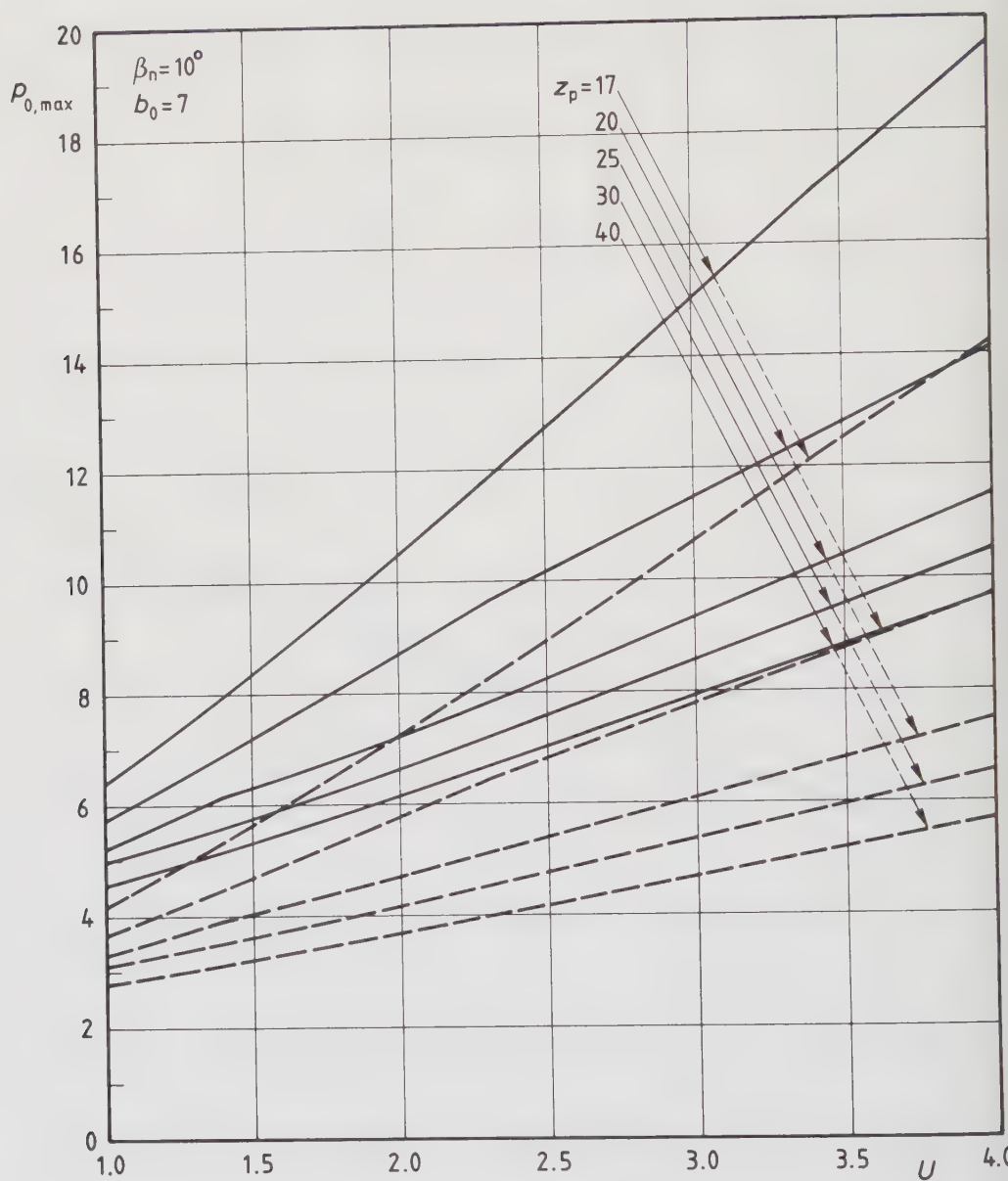


Fig. 5.4. Helical gear contact pressures for gears without tip relief (unbroken lines), compared with contact pressures from uniform load distributions (dashed lines)

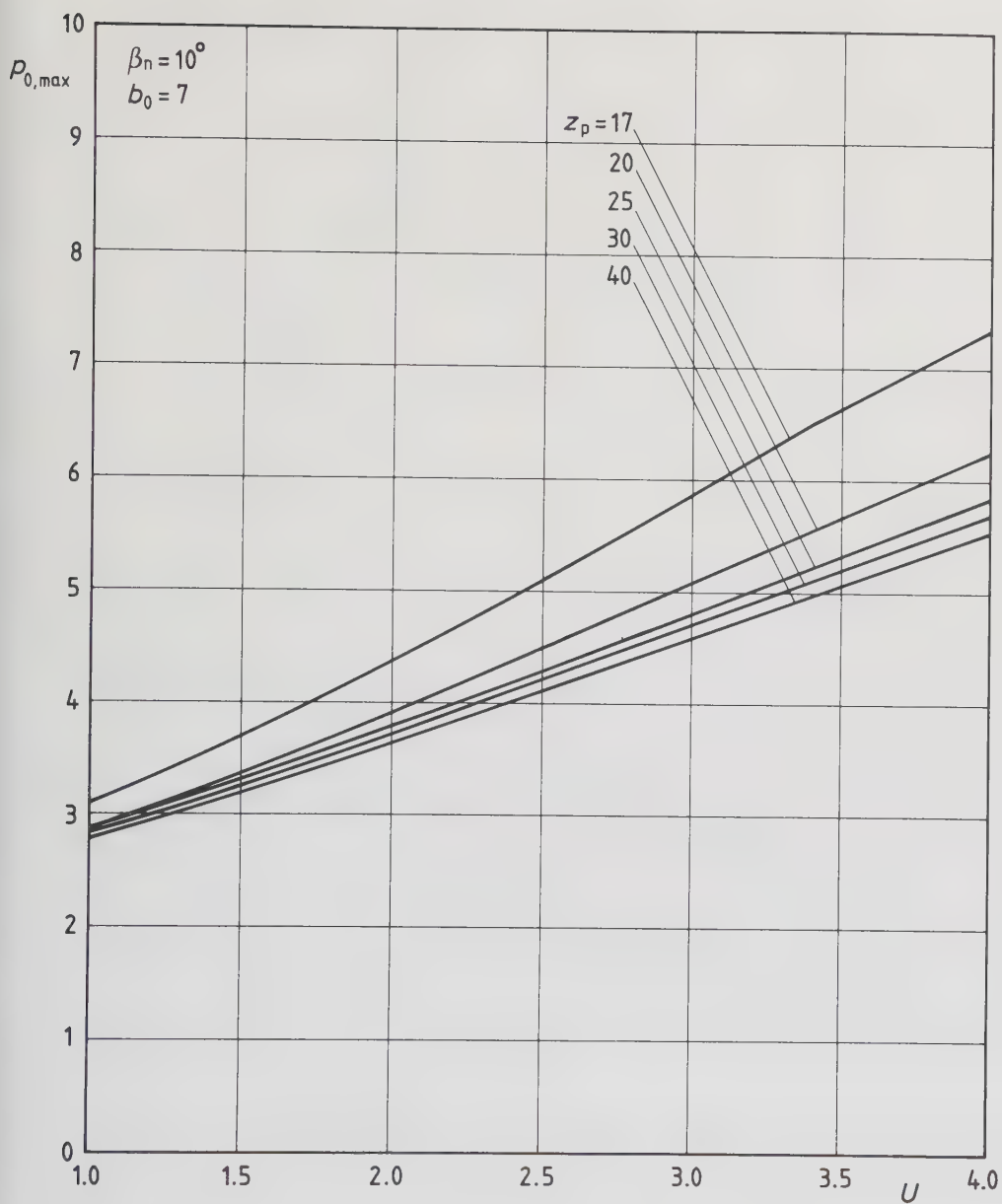


Fig. 5.5. Helical gear contact pressures for gears with tip relief

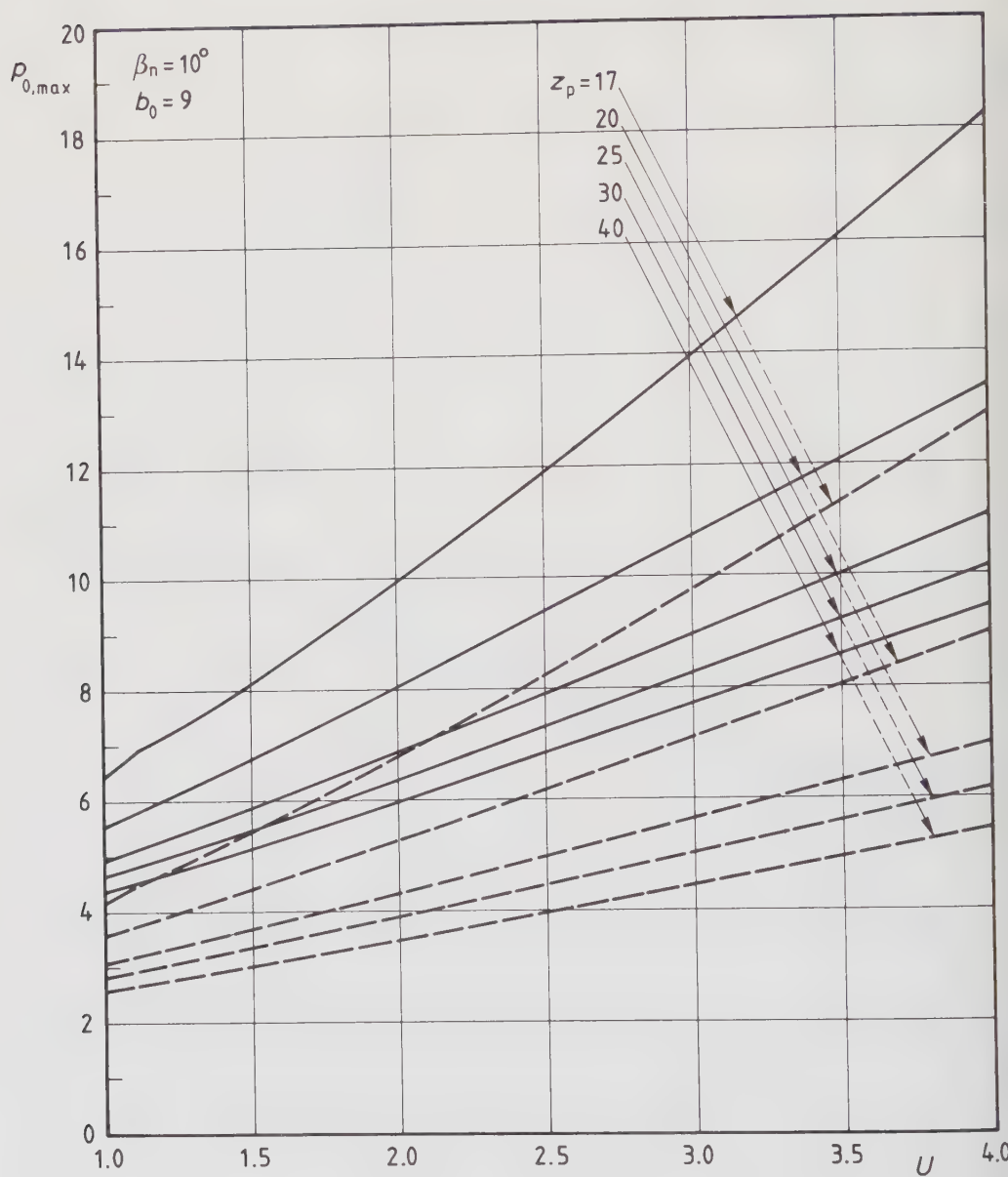


Fig. 5.6. Helical gear contact pressures for gears without tip relief (unbroken lines), compared with contact pressures from uniform load distributions (dashed lines)

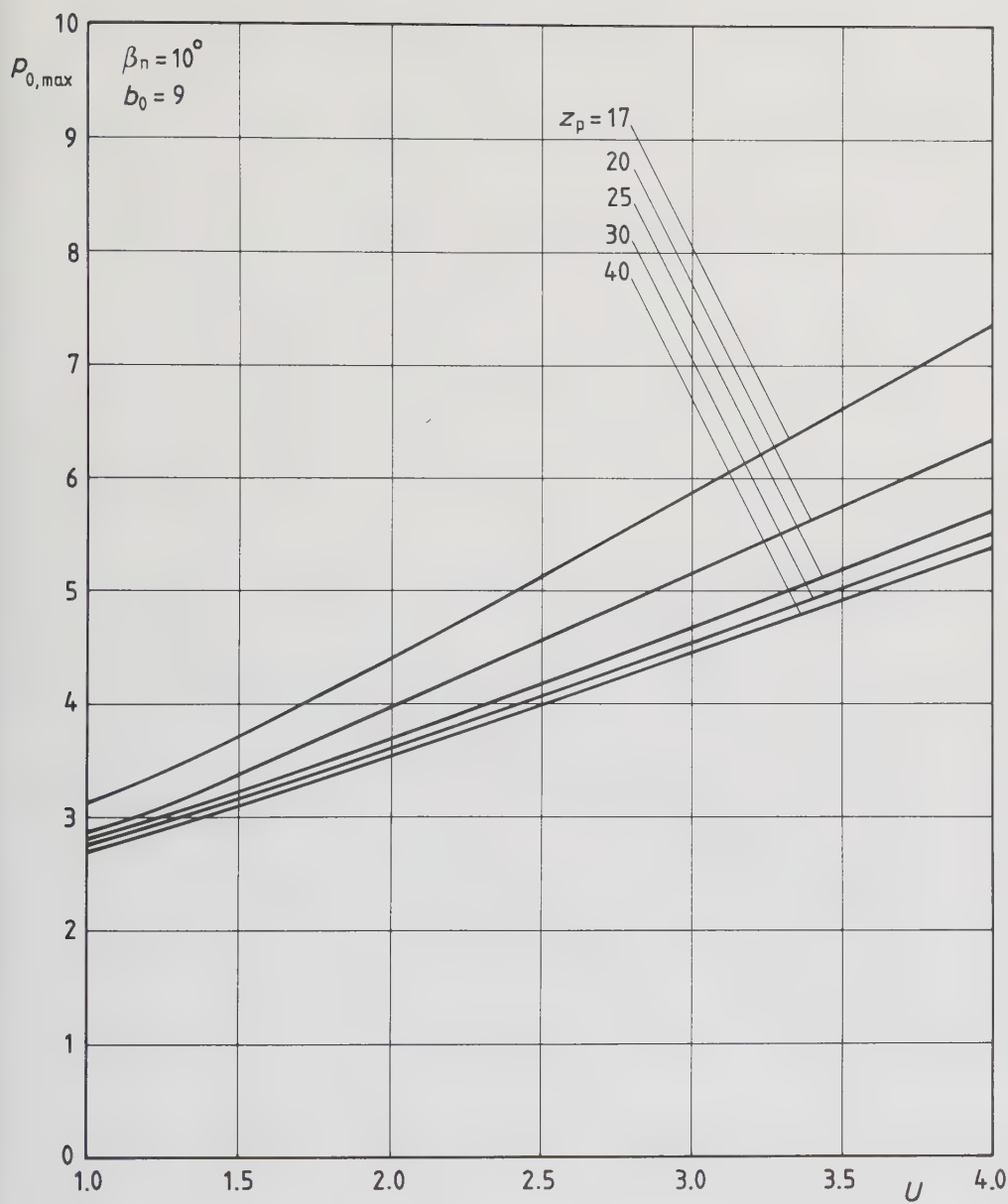


Fig. 5.7. Helical gear contact pressures for gears with tip relief

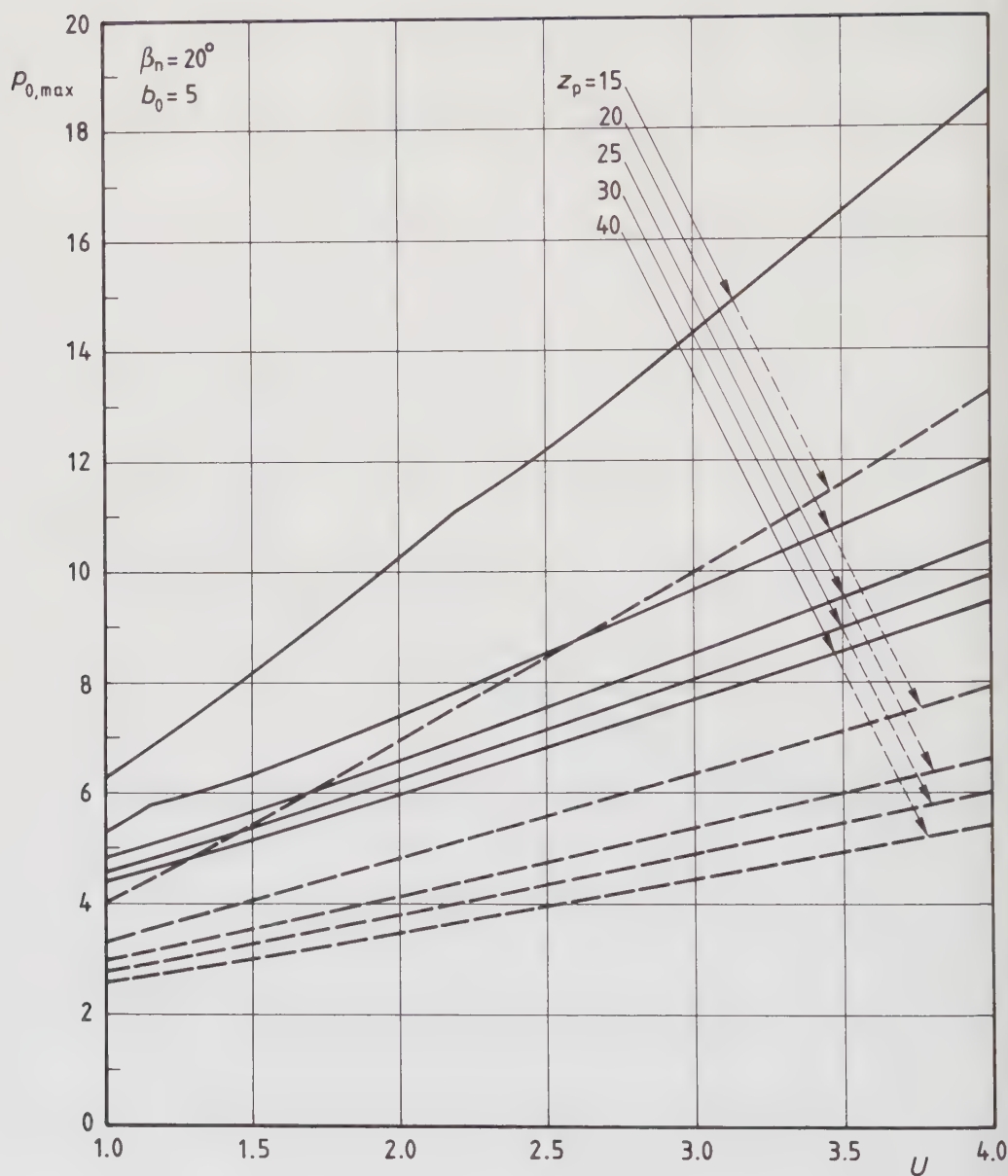


Fig. 5.8. Helical gear contact pressures for gears without tip relief (unbroken lines), compared with contact pressures from uniform load distributions (dashed lines)

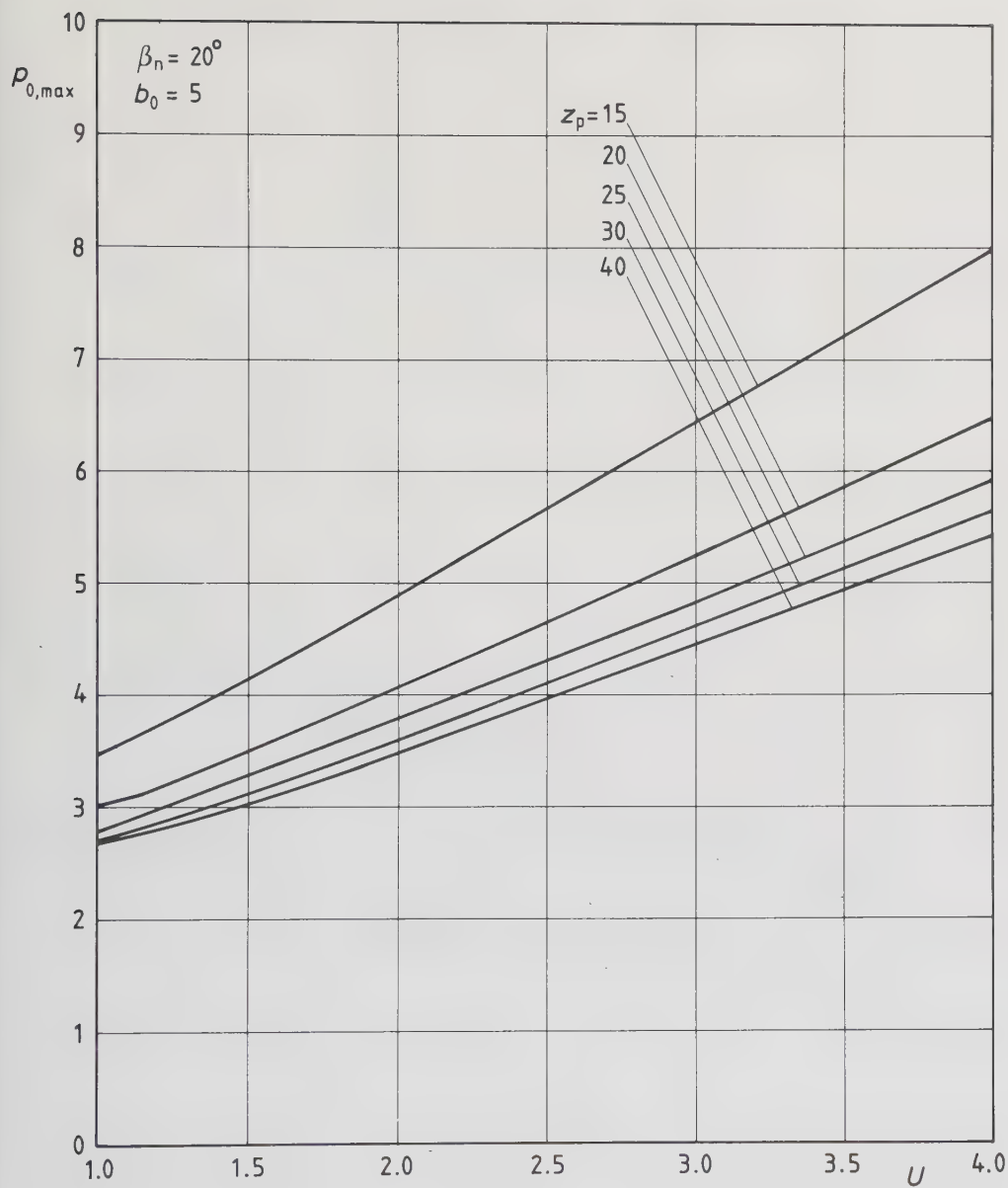


Fig. 5.9. Helical gear contact pressures for gears with tip relief

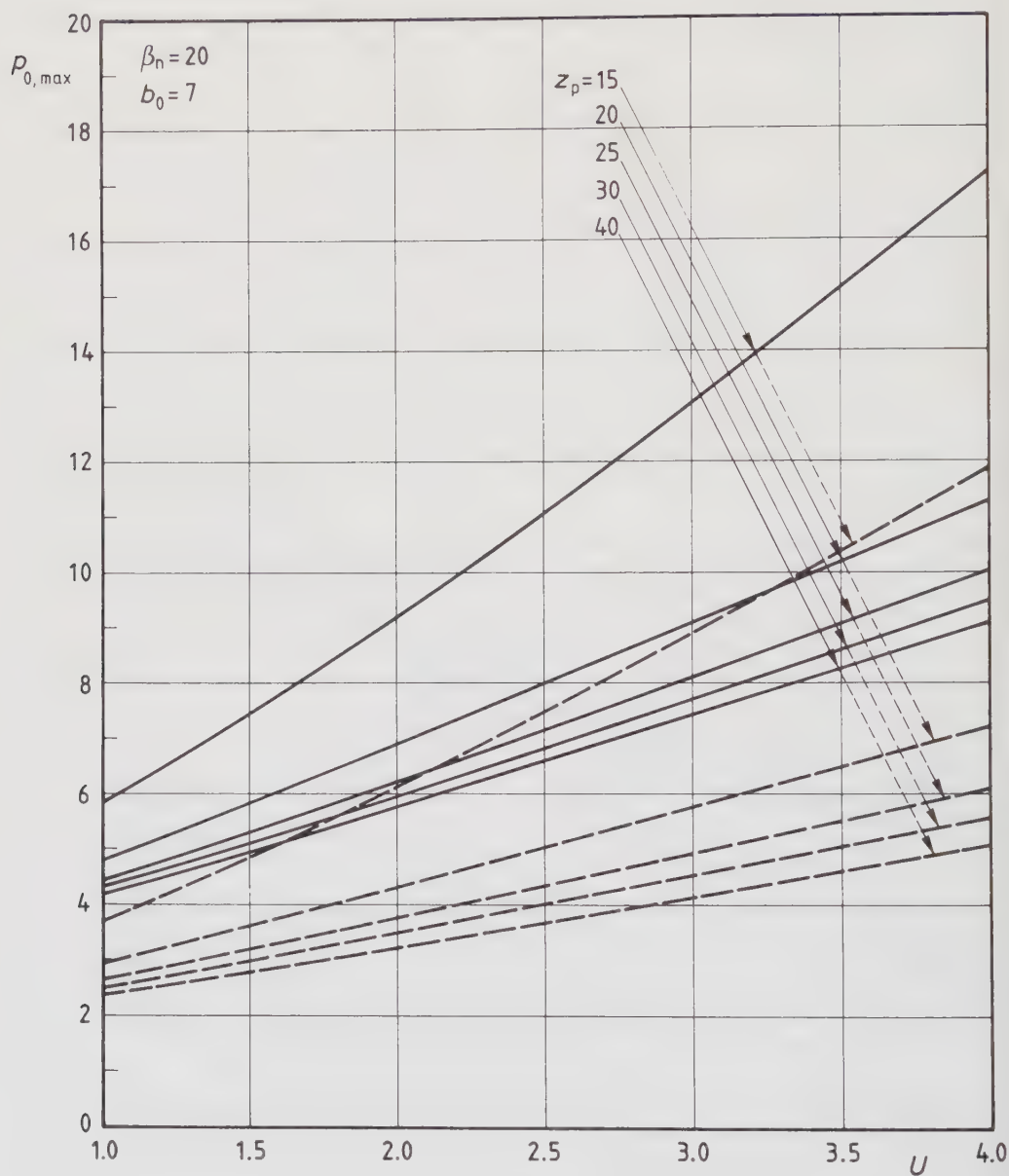


Fig. 5.10. Helical gear contact pressures for gears without tip relief (unbroken lines), compared with contact pressures from uniform load distributions (dashed lines)

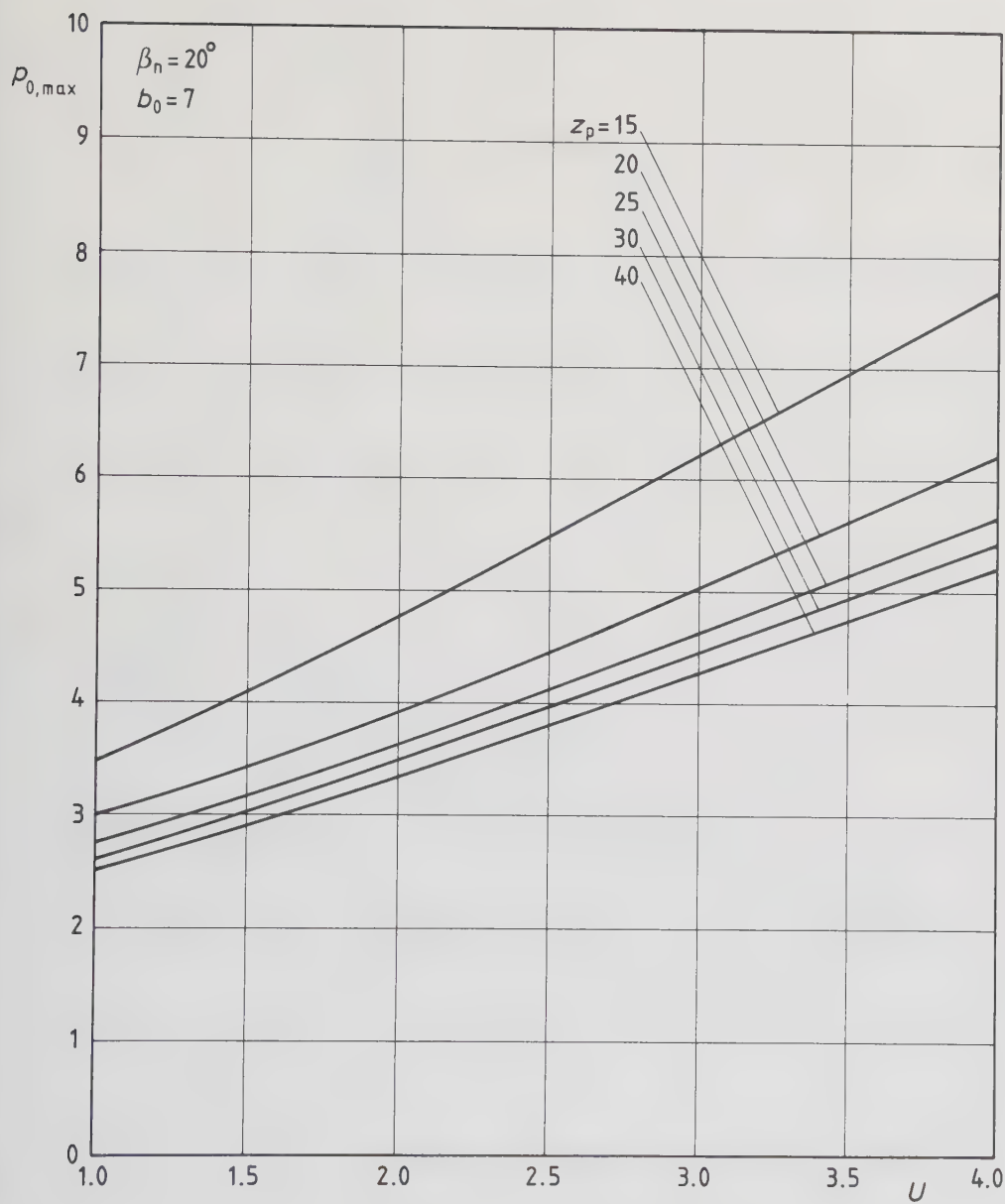


Fig. 5.11. Helical gear contact pressures for gears with tip relief

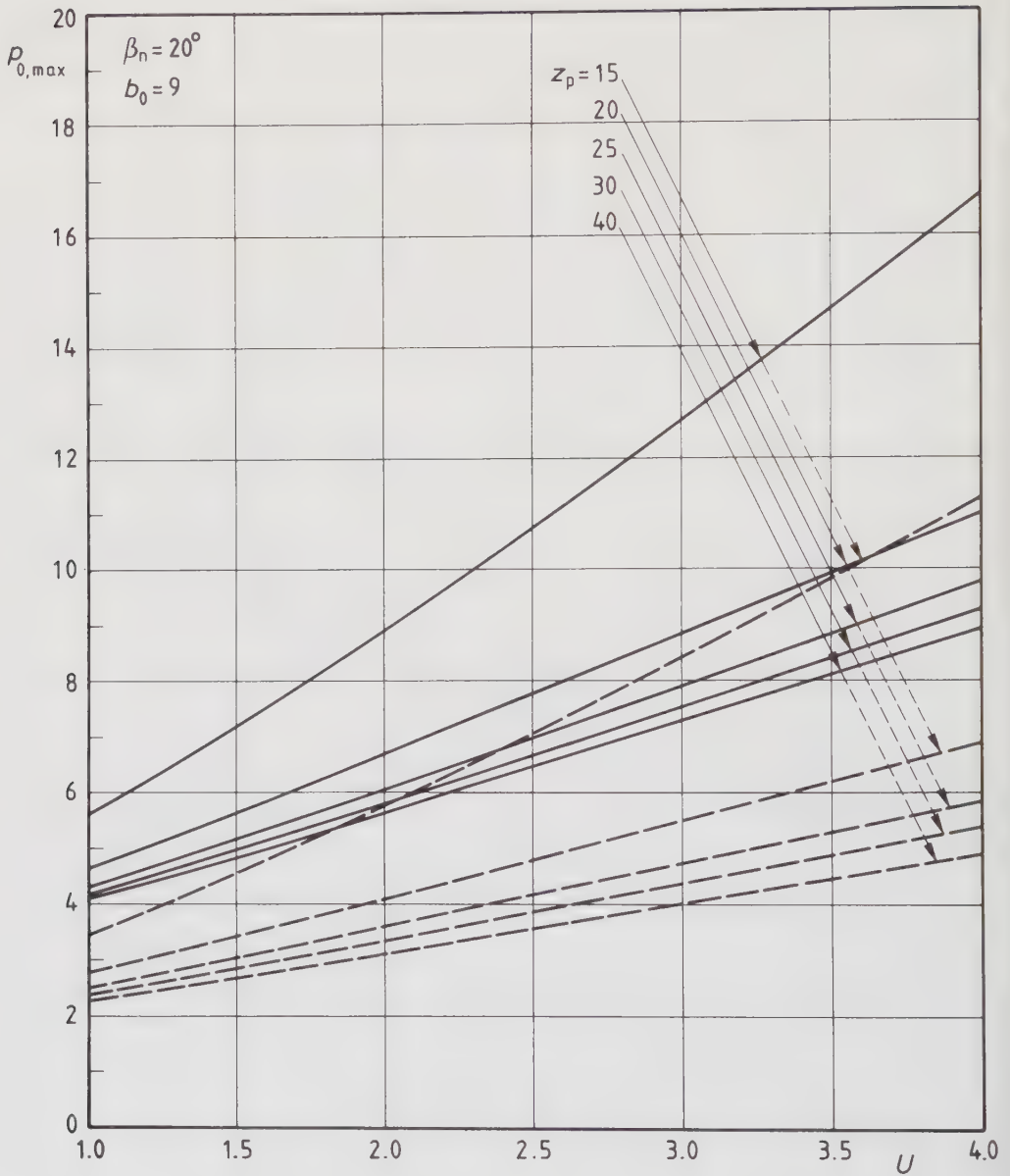


Fig. 5.12. Helical gear contact pressures for gears without tip relief (unbroken lines), compared with contact pressures from uniform load distributions (dashed lines)

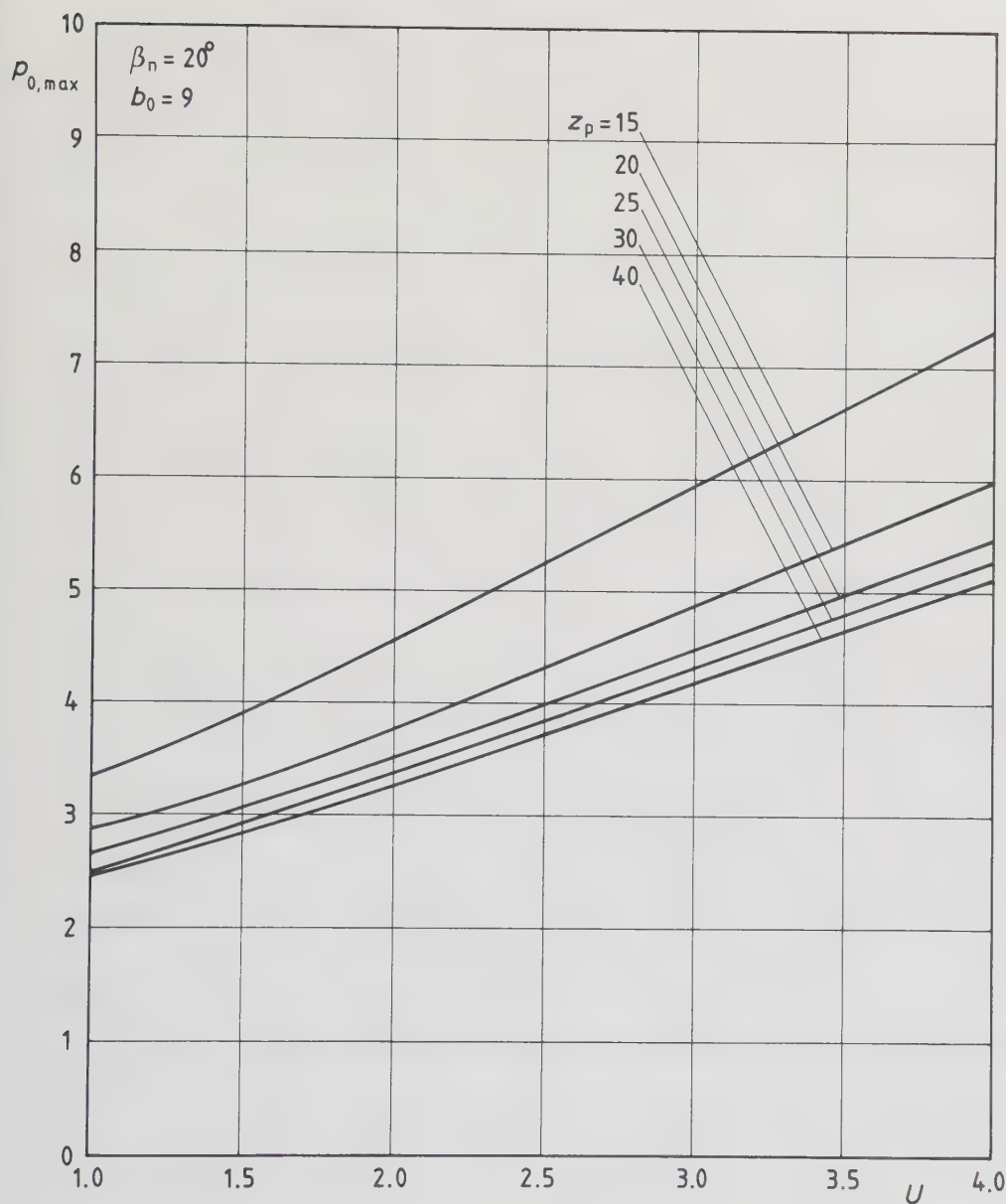


Fig. 5.13. Helical gear contact pressures for gears with tip relief

6. Gear Bending Stresses

The stresses in the fillet at the tensile and compression sides were calculated by the finite element program at the same time as the bending displacement, but since the load distribution along the contact lines was not known at this stage of the determination, unit forces were applied in the nodal points. To obtain the stresses in the finest element division at the fillet, the unit forces were applied at the left flank of the tooth in Fig. 4.2 when the compression stresses were to be determined.

By replacing the load at a contact line with discrete forces in the transverse planes, in which the nodal unit forces were applied in the calculation carried out by the finite element program, in such a way that the forces are in equilibrium with the distributed forces at the contact line, for instance using Eq. 4.14, and by summing the stresses for the unit force multiplied by the current discrete force, the total stresses are obtained for this load case.

The stresses of the pinion and the gear have been calculated for $b_o = 5$, $\beta_n = 0, 10^\circ$ and 20° , $z_p = z_{min}, 20, 25, 30$ and 40 and $z_g = z_p, \dots, 80$ and, if $80/z_p < 4$, 160 for about 20 positions of the load.

Fig. 6.1 to 6.18 show how the maximum stress at the fillet varies with the position of the load and how the stresses are distributed in the pinion, where all maximum stress occurs, in the transverse plane involving the stress in the force position giving maximum stress for some examples of gear sets.

In all cases maximum stress occurs in the force position at the greatest distance from the gear centre when the total contact length is minimum, i.e. one base pitch from the point A_g' when $\epsilon < 2$ and one base pitch from the point A_p when $\epsilon > 2$. The load at this position when $\epsilon < 2$ is independent of any tip relief of the gears, but the load increases with the tip relief when $\epsilon > 2$ thus making the stress increase.

The maximum stress occurs in the axial position where the current contact line is at the greatest distance from the gear centre and the stress is not influenced

by the end effects of the boundary $A_p' A_g'$. The stress contribution from load positions, at the same contact line, near the boundary $A_p A_g$ is at this position very small, not only because of the relatively great distance to the point where maximum stress occurs but also because of the lower height position at the tooth. This is used when calculating the stresses for gears with face widths other than $b_o = 5$, with the assumption that the maximum stress, in a load position between the points A_p' and A_g where the contact reaches over the whole face width, per equivalent uniform load intensity Q/L is independent of the face width b where Q is the load at the contact line, with the length L , at the current load position. This assumption makes the maximum stresses independent of b when $\epsilon < 2$ but when $\epsilon > 2$ the load Q decreases with increasing contact ratio ϵ and also the position of the load, giving maximum stress, relative to the point A_p' is changed.

Wennerström (10) has given the basic rule for comparing two different gears which was based upon equal gear ratio U , equal torque on the pinion M_p , equal width of the gear face b and equal centre distance a . Introducing a non-dimensional stress based on this rule and using Eq. 5.2 we obtain

$$\sigma_o = \frac{|\sigma| a^2 b}{M_p} = \frac{|\sigma| m_n^2}{F} \cdot \frac{z_p b_o (1 + U)^2}{2 \cos^2 \beta_n \cos \alpha_n} \cdot \frac{\cos^2 \alpha_t}{\cos^2 \alpha} \quad (6.1)$$

In appendix A3 this non-dimensional stress is given for all the calculated gear sets whether or not the gears are relieved, but in the graphic representation in Fig. 6.19 to 6.32 the factor $\sigma_o / (z_p (1 + U)^2)$ is plotted since this factor varies very little with the gear ratio, a fact also mentioned by Wennerström.

When the gears are relieved the same size of tip relief is used as in section 5.

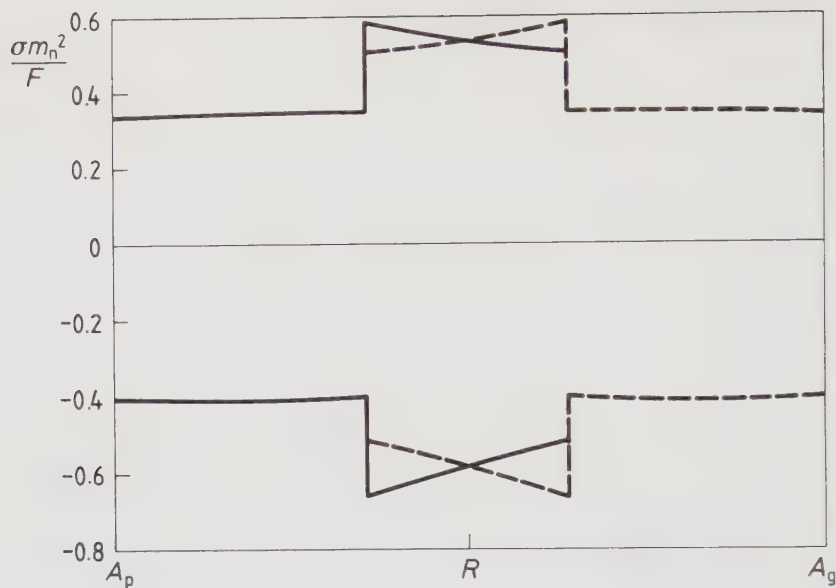


Fig. 6.1. Maximum fillet stresses at the pinion and the gear as function of the position of the load at the path of contact

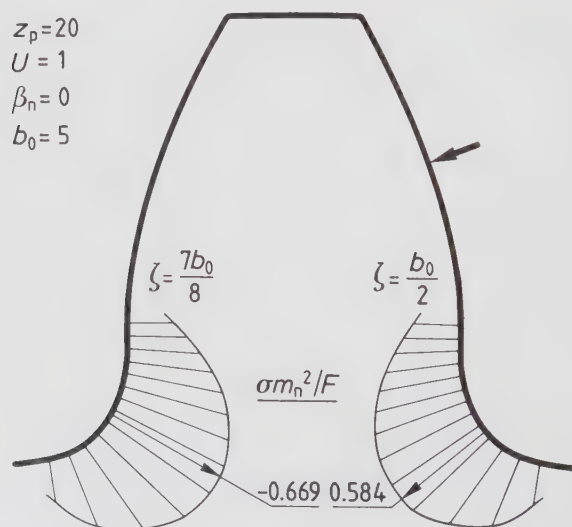


Fig. 6.2. Fillet stresses at the pinion in the transverse planes at the position of the force where the stress is greatest

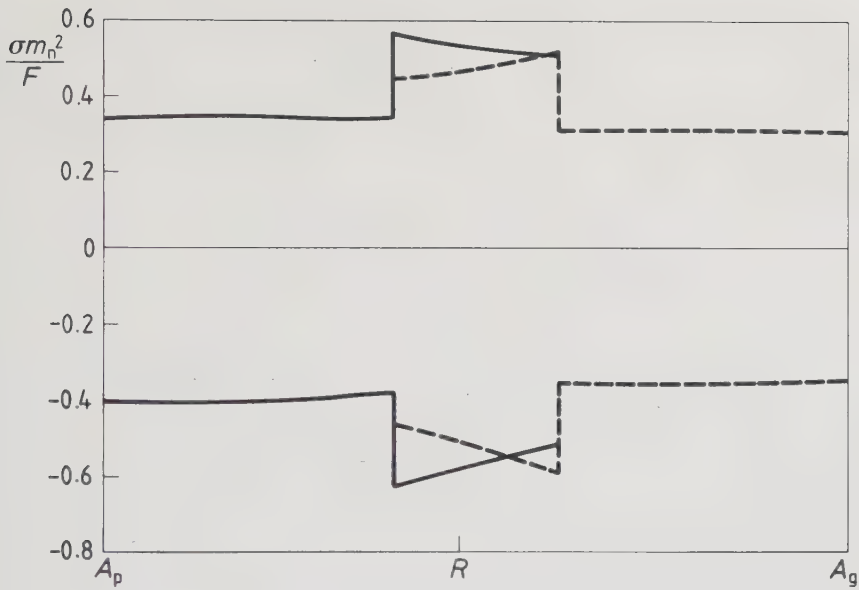


Fig. 6.3. Maximum fillet stresses at the pinion and the gear as function of the position of the load at the path of contact

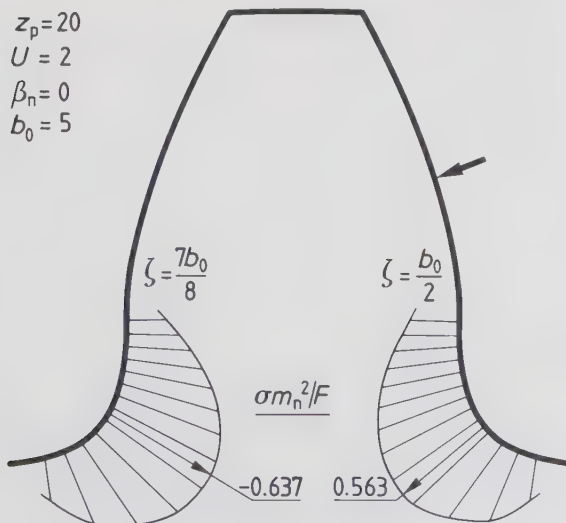


Fig. 6.4. Fillet stresses at the pinion in the transverse planes at the position of the force where the stress is greatest

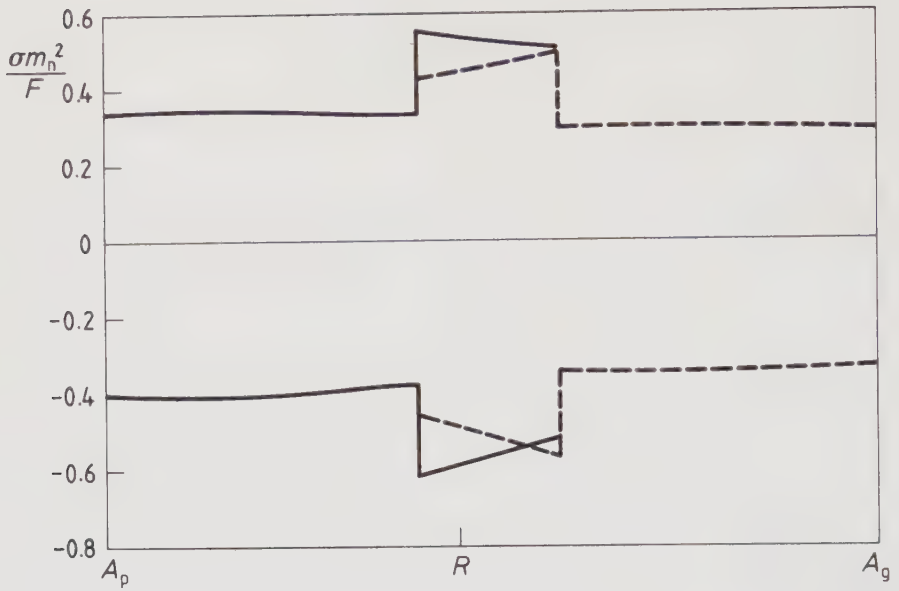


Fig. 6.5. Maximum fillet stresses at the pinion and the gear as function of the position of the load at the path of contact

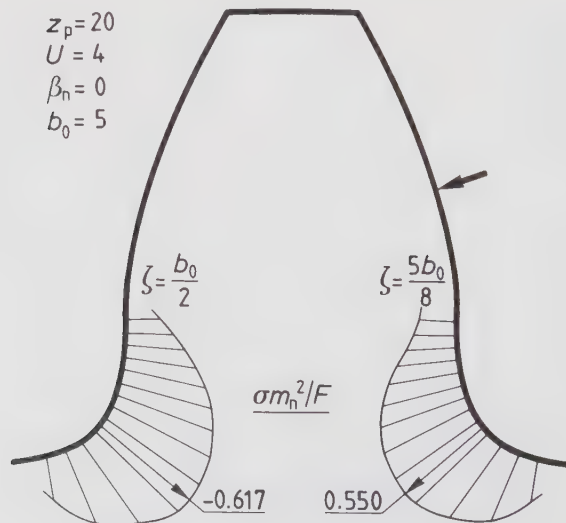


Fig. 6.6. Fillet stresses at the pinion in the transverse planes at the position of the force where the stress is greatest

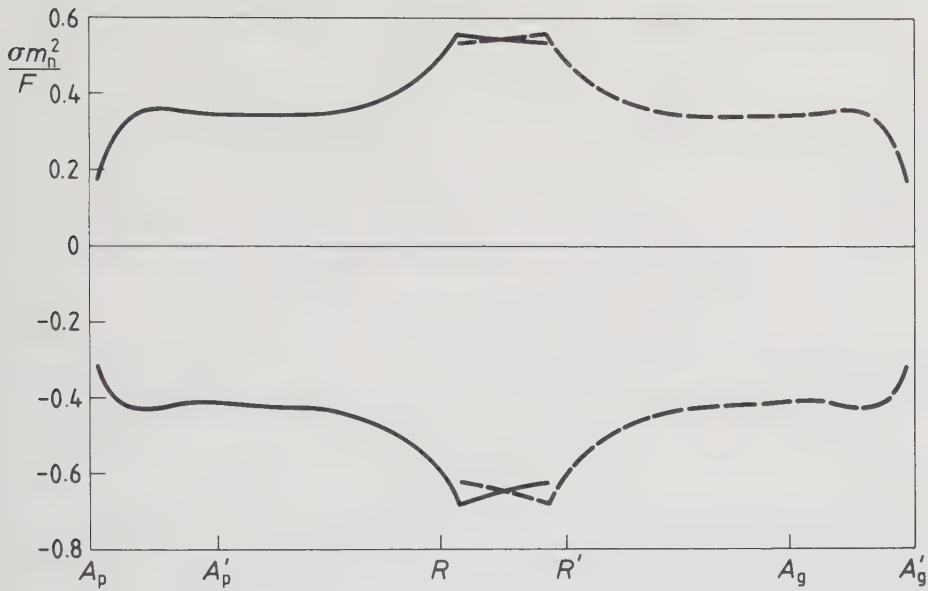


Fig. 6.7. Maximum fillet stresses at the pinion and the gear as function of the position of the load at the path of contact

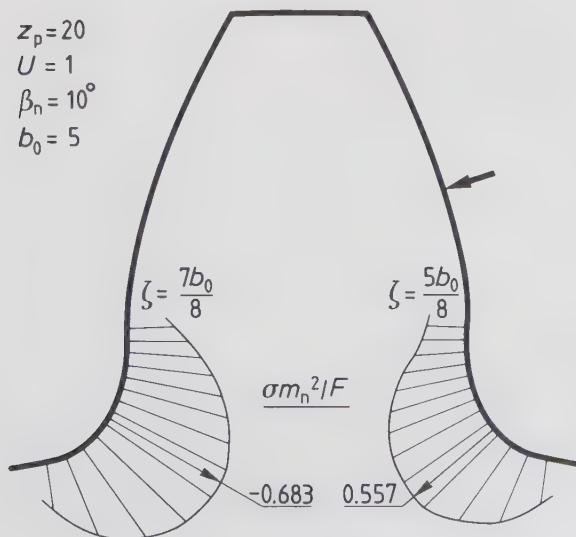


Fig. 6.8. Fillet stresses at the pinion in the transverse planes at the position of the force where the stress is greatest

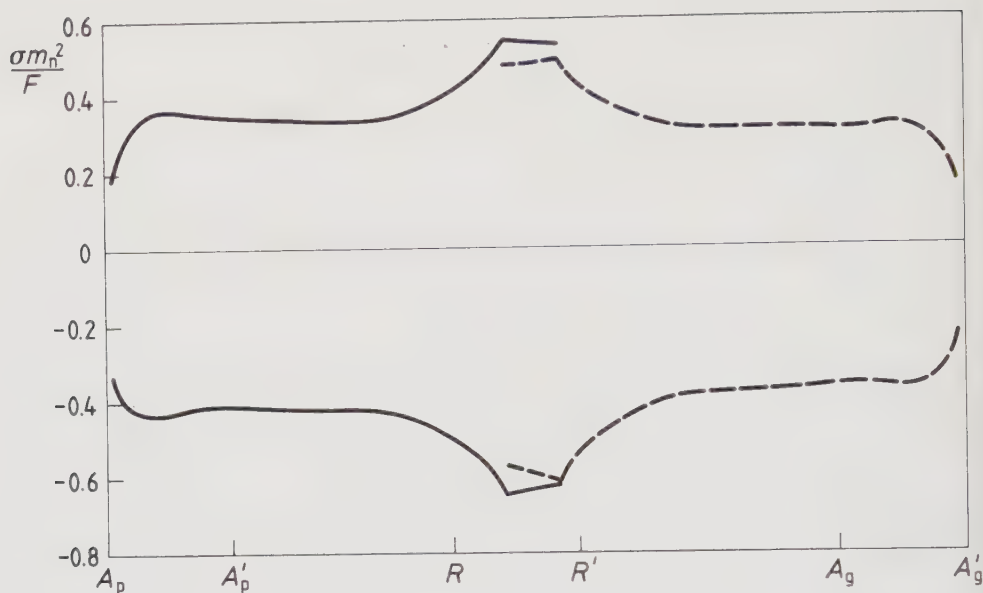


Fig. 6.9. Maximum fillet stresses at the pinion and the gear as function of the position of the load at the path of contact

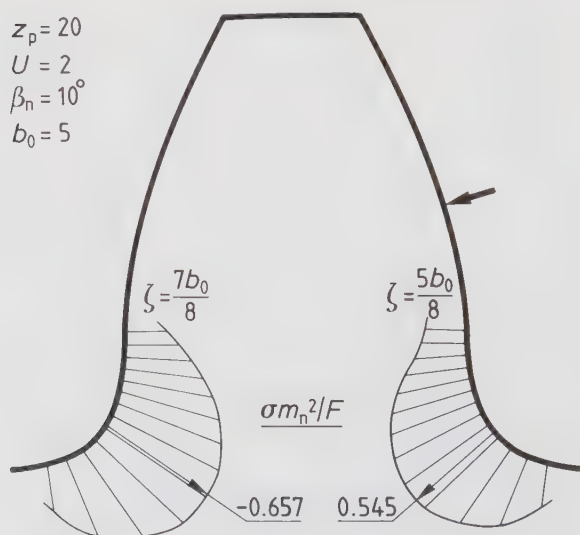


Fig. 6.10. Fillet stresses at the pinion in the transverse planes at the position of the force where the stress is greatest

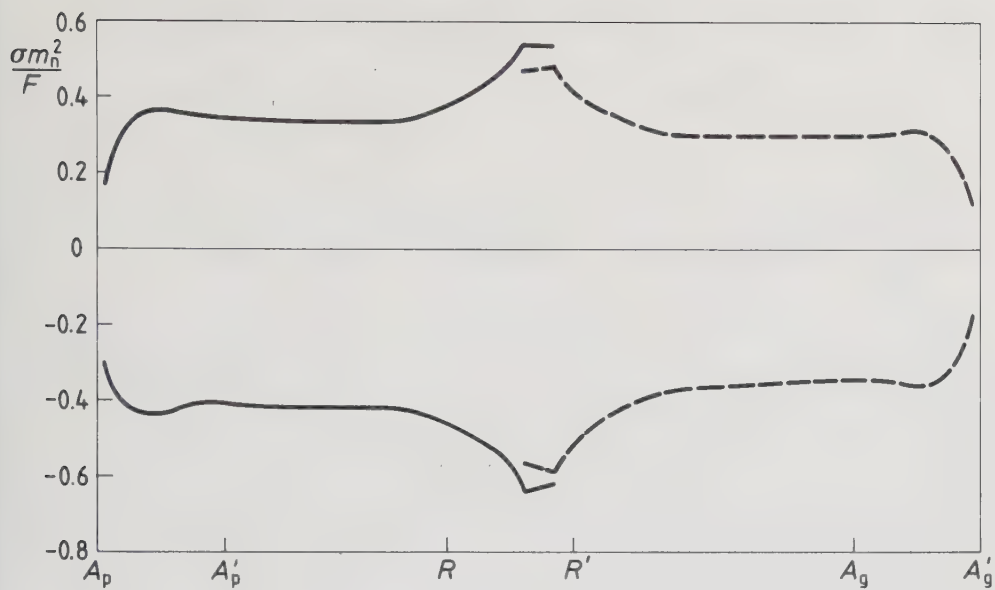


Fig. 6.11. Maximum fillet stresses at the pinion and the gear as function of the position of the load at the path of contact

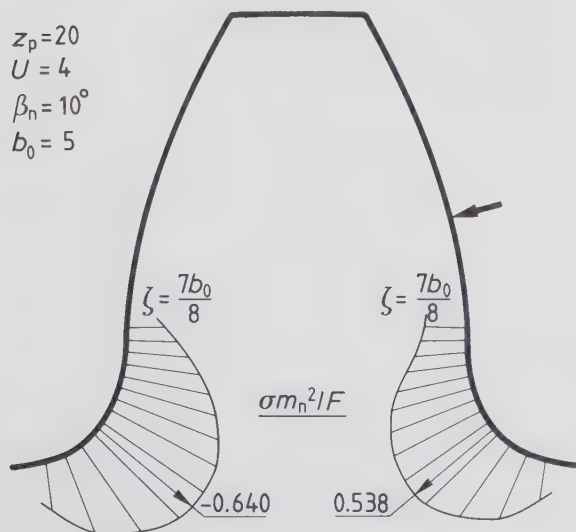


Fig. 6.12. Fillet stresses at the pinion in the transverse planes at the position of the force where the stress is greatest

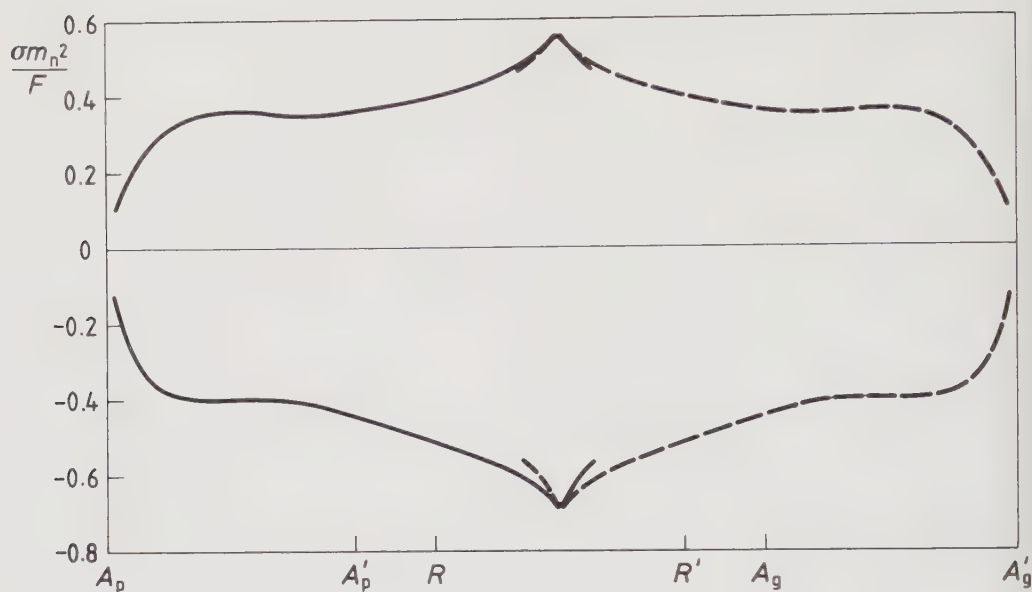


Fig. 6.13. Maximum fillet stresses at the pinion and the gear as function of the position of the load at the path of contact

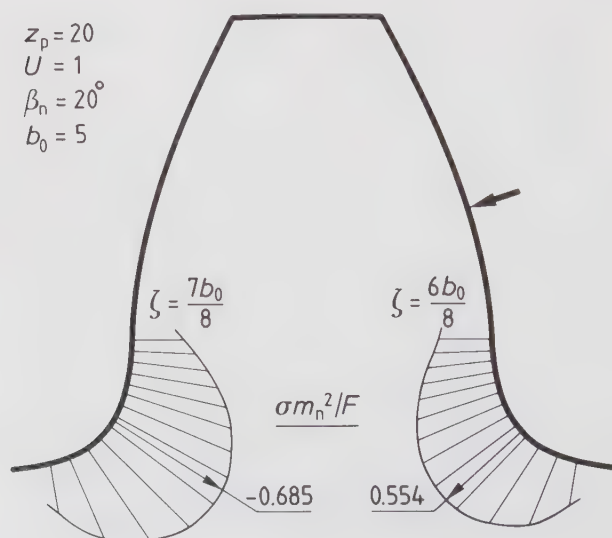


Fig. 6.14. Fillet stresses at the pinion in the transverse planes at the position of the force where the stress is greatest

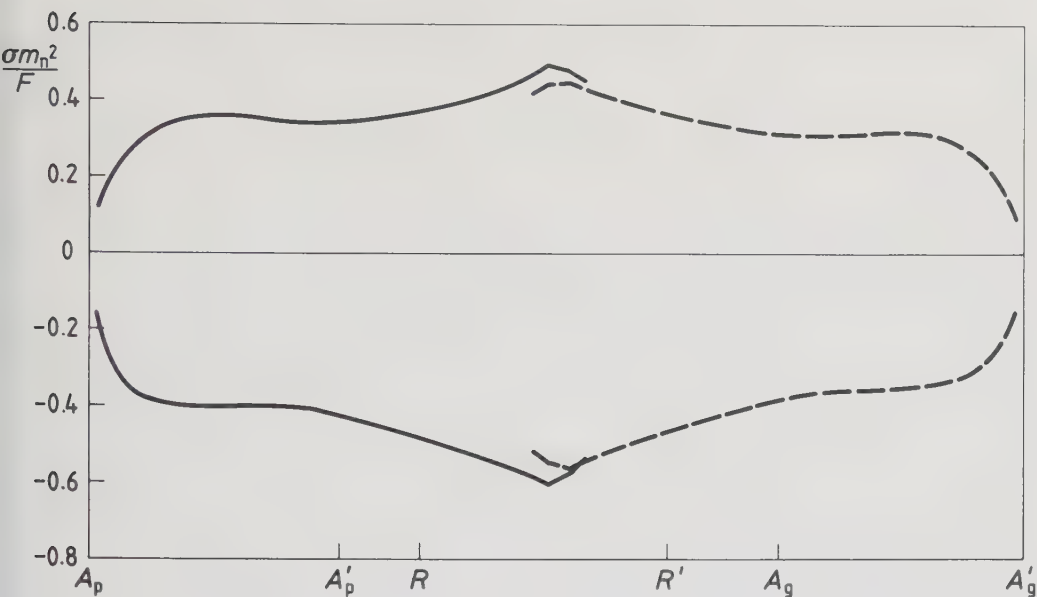


Fig. 6.15. Maximum fillet stresses at the pinion and the gear as function of the position of the load at the path of contact

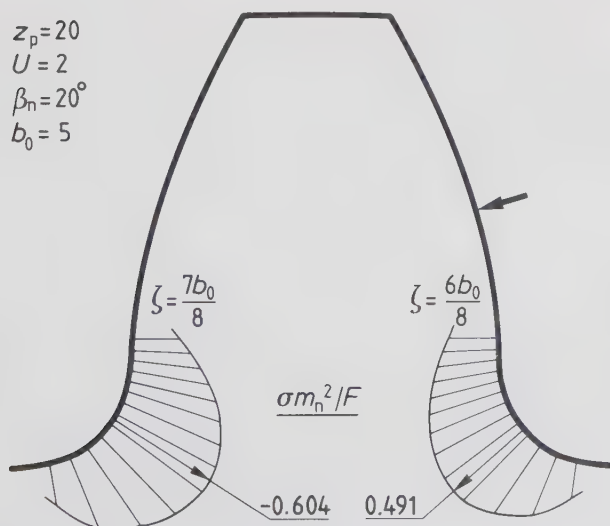


Fig. 6.16. Fillet stresses at the pinion in the transverse planes at the position of the force where the stress is greatest

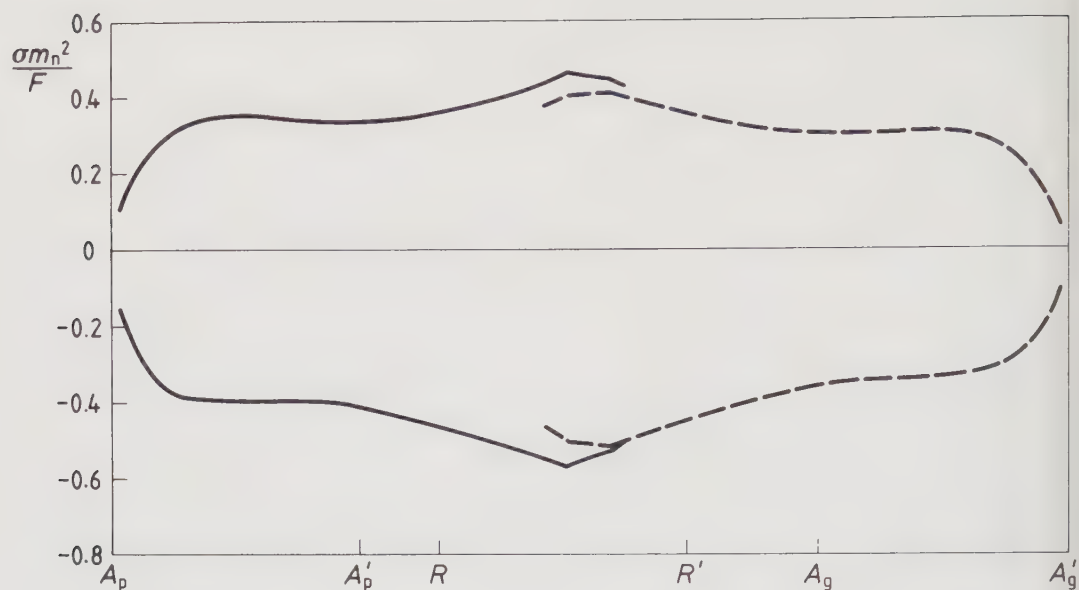


Fig. 6.17. Maximum fillet stresses at the pinion and the gear as function of the position of the load at the path of contact

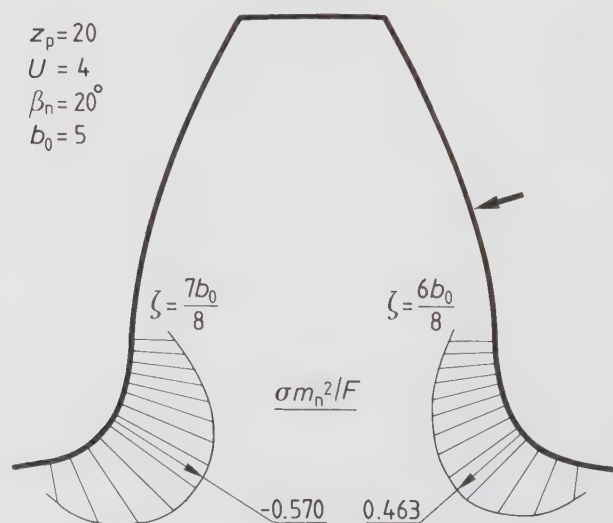


Fig. 6.18. Fillet stresses at the pinion in the transverse planes at the position of the force where the stress is greatest

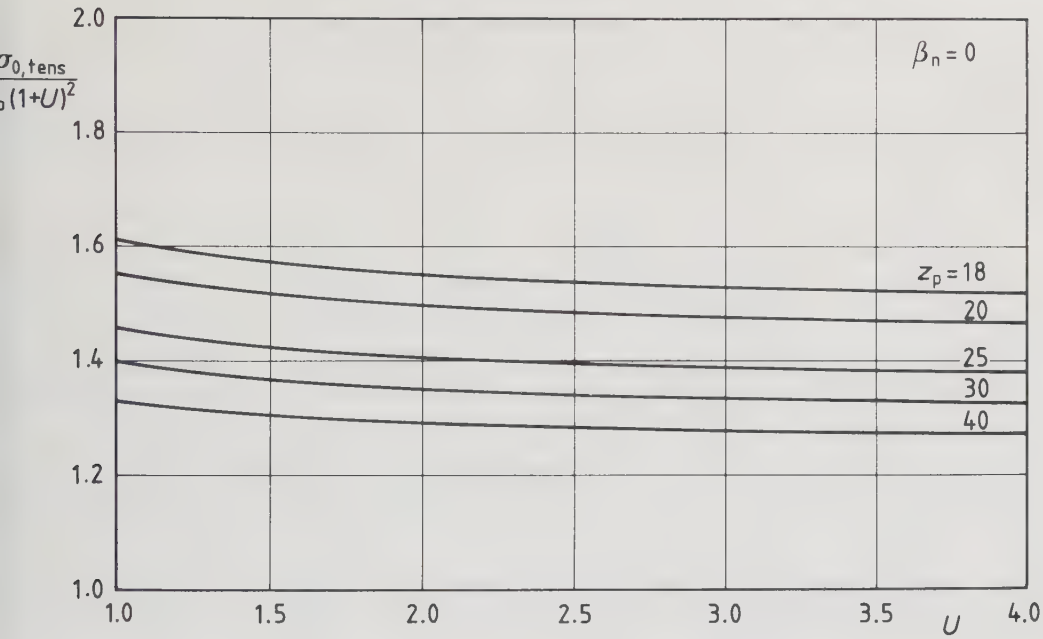


Fig. 6.19. Non-dimensional tensile stresses

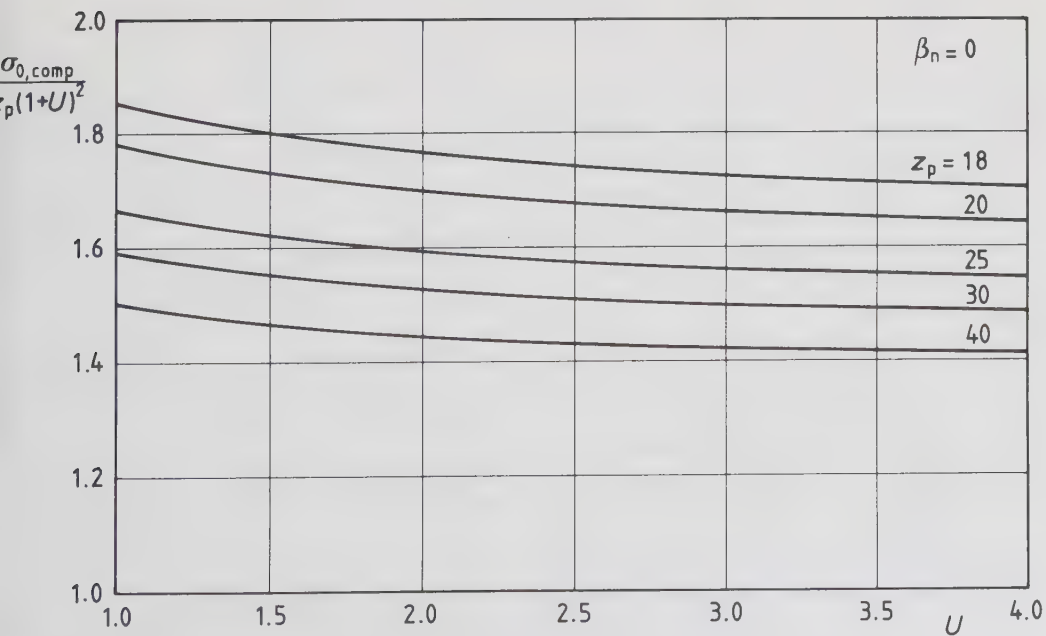


Fig. 6.20. Non-dimensional compression stresses

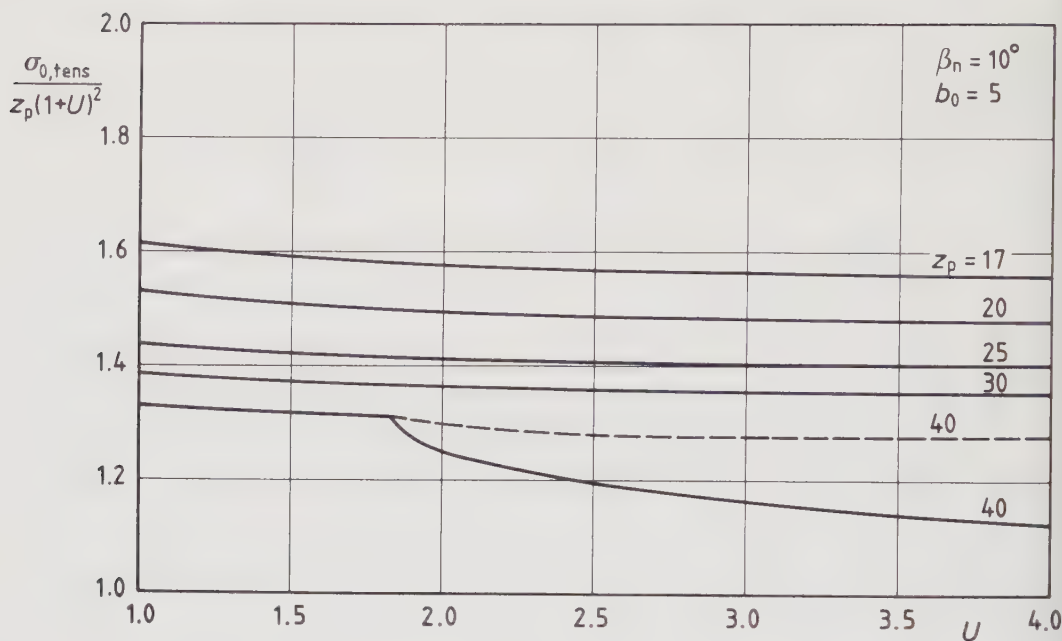


Fig. 6.21. Non-dimensional tensile stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

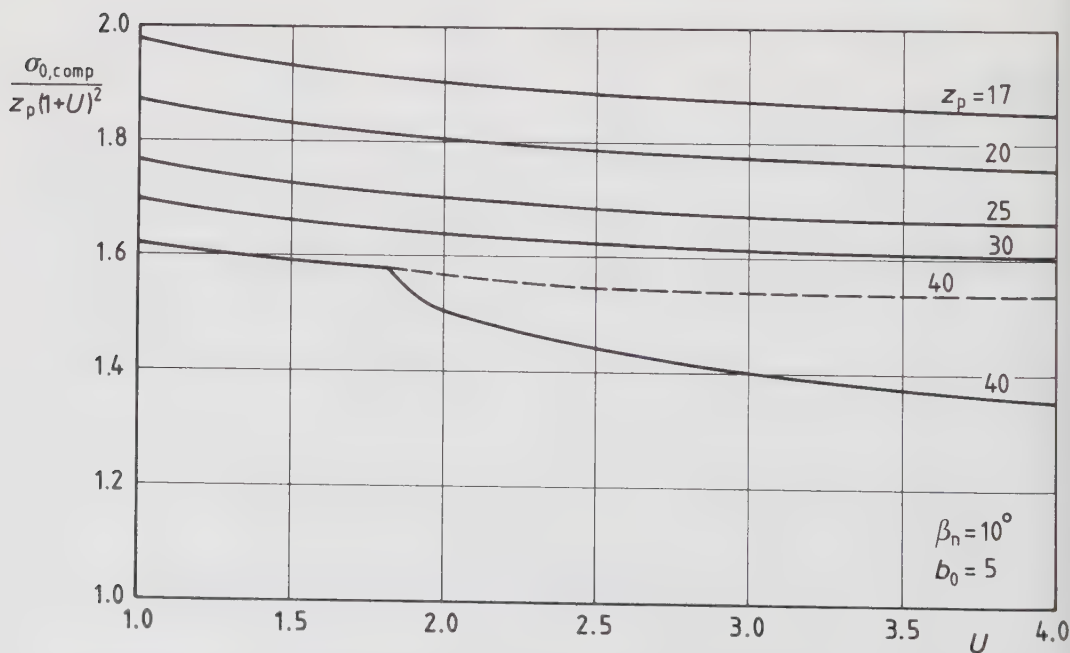


Fig. 6.22. Non-dimensional compression stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

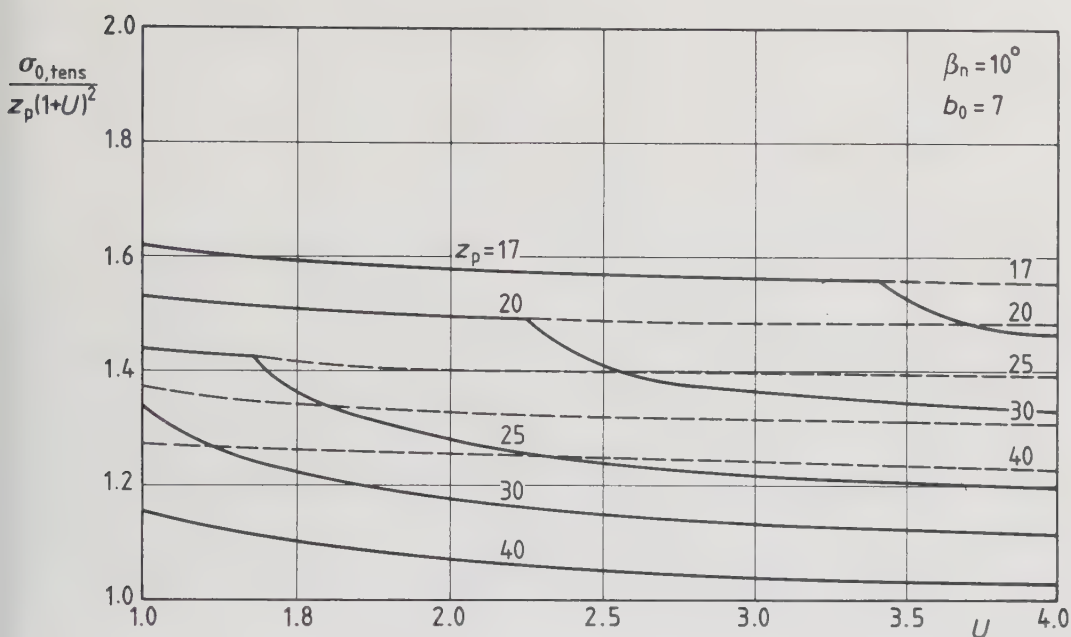


Fig. 6.23. Non-dimensional tensile stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

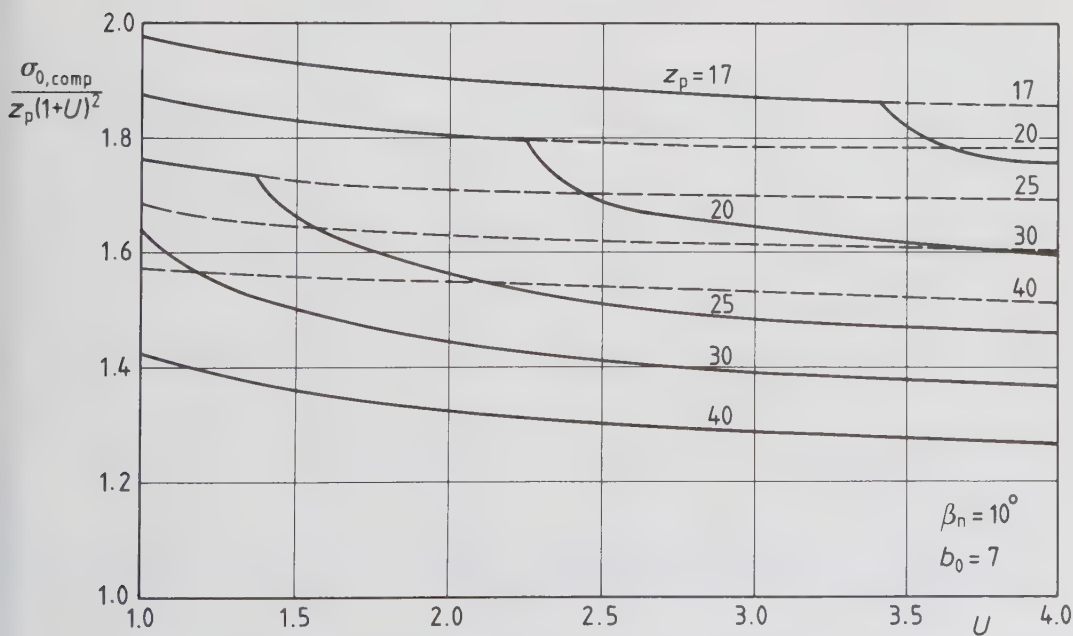


Fig. 6.24. Non-dimensional compression stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

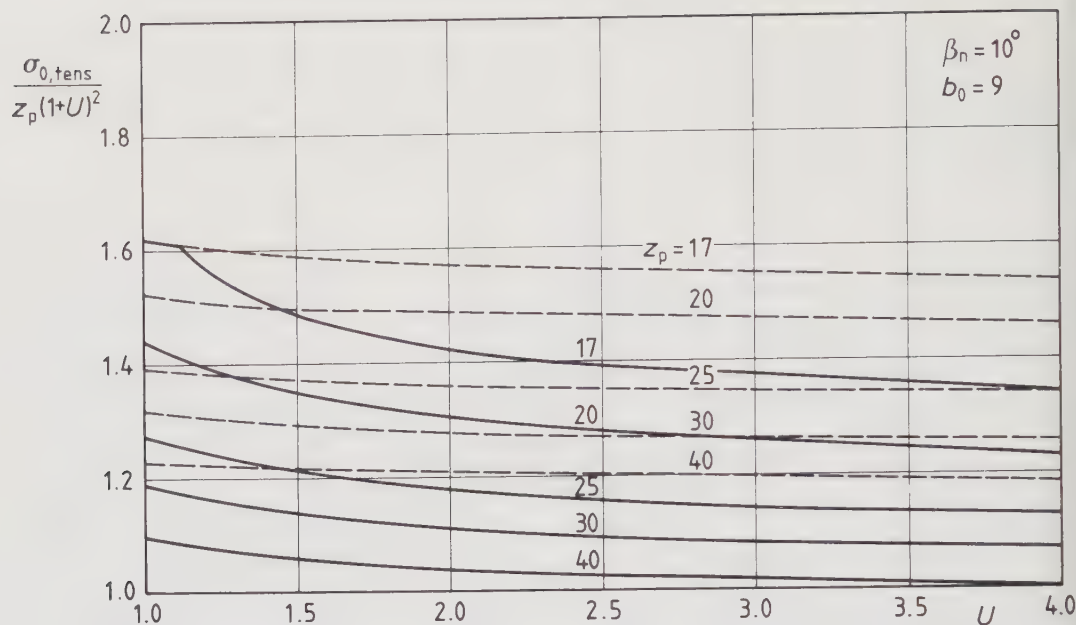


Fig. 6.25. Non-dimensional tensile stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

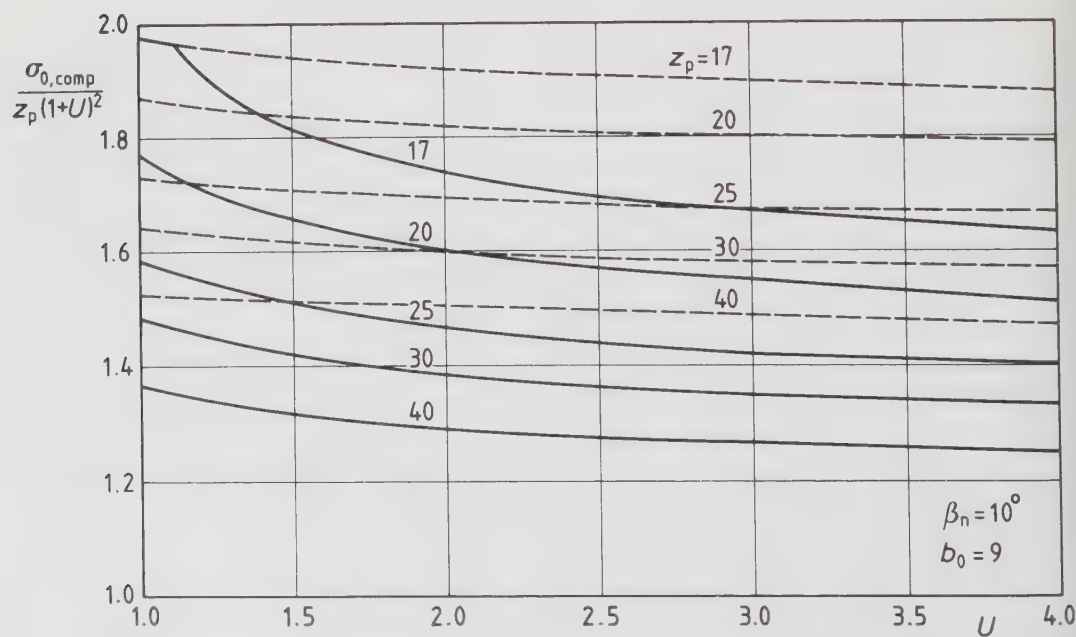


Fig. 6.26. Non-dimensional compression stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

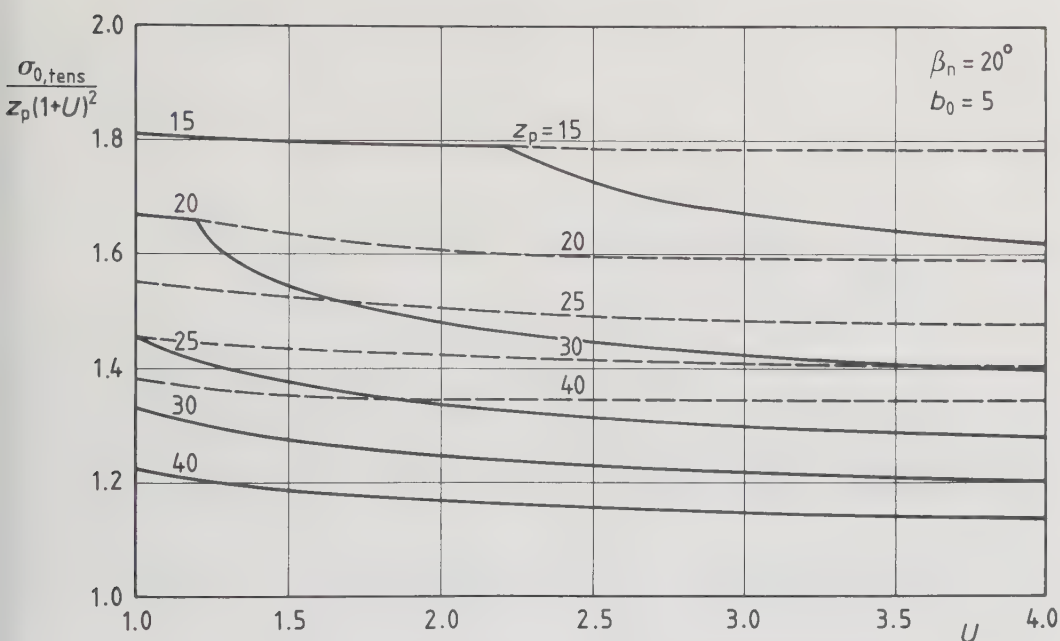


Fig. 6.27. Non-dimensional tensile stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

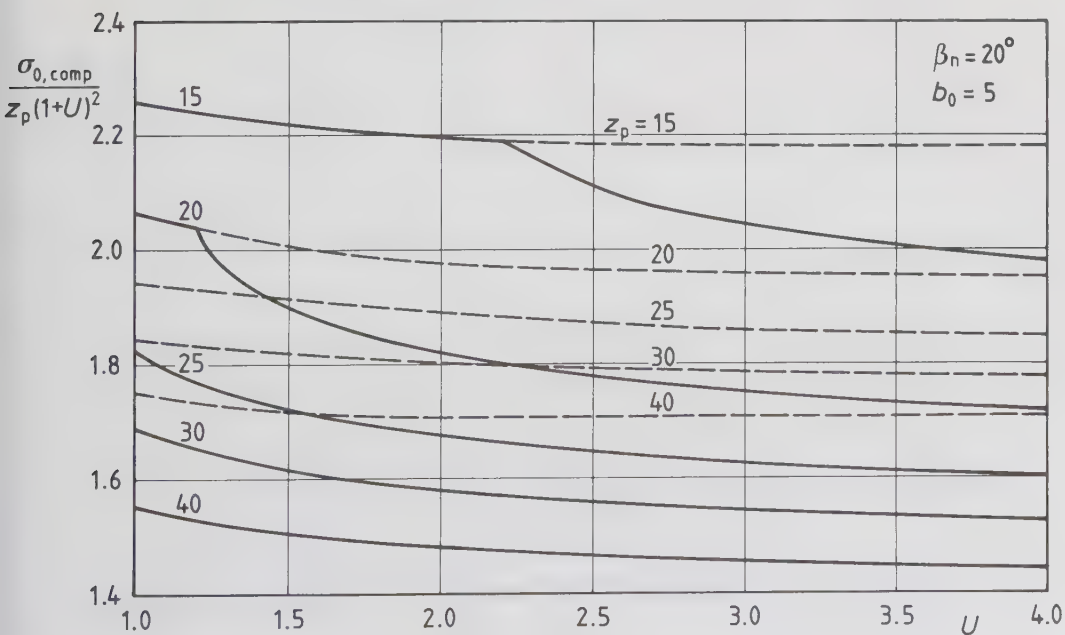


Fig. 6.28. Non-dimensional compression stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

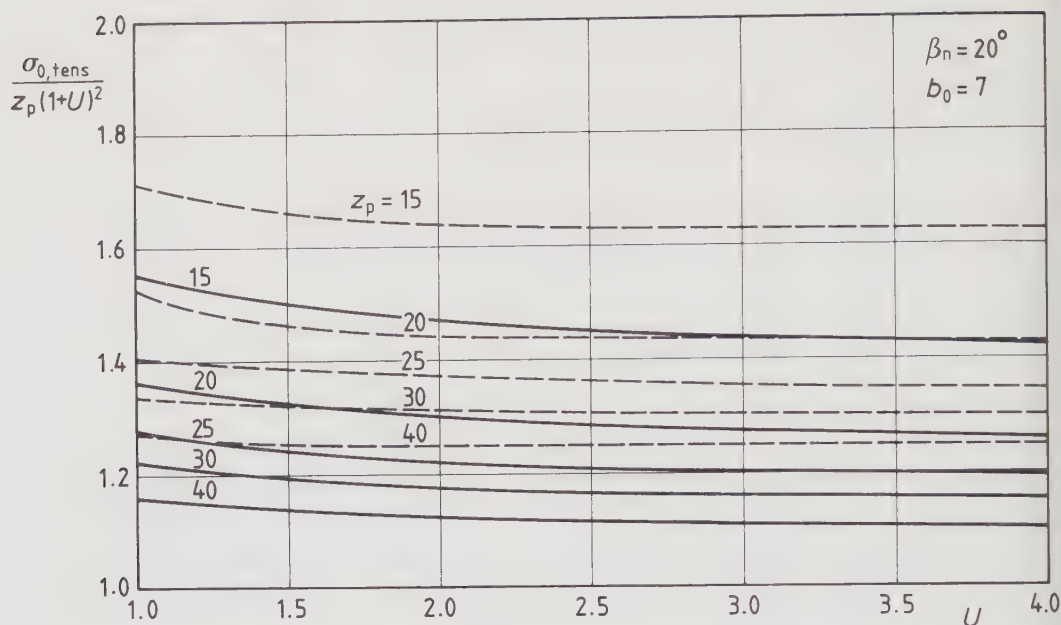


Fig. 6.29. Non-dimensional tensile stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

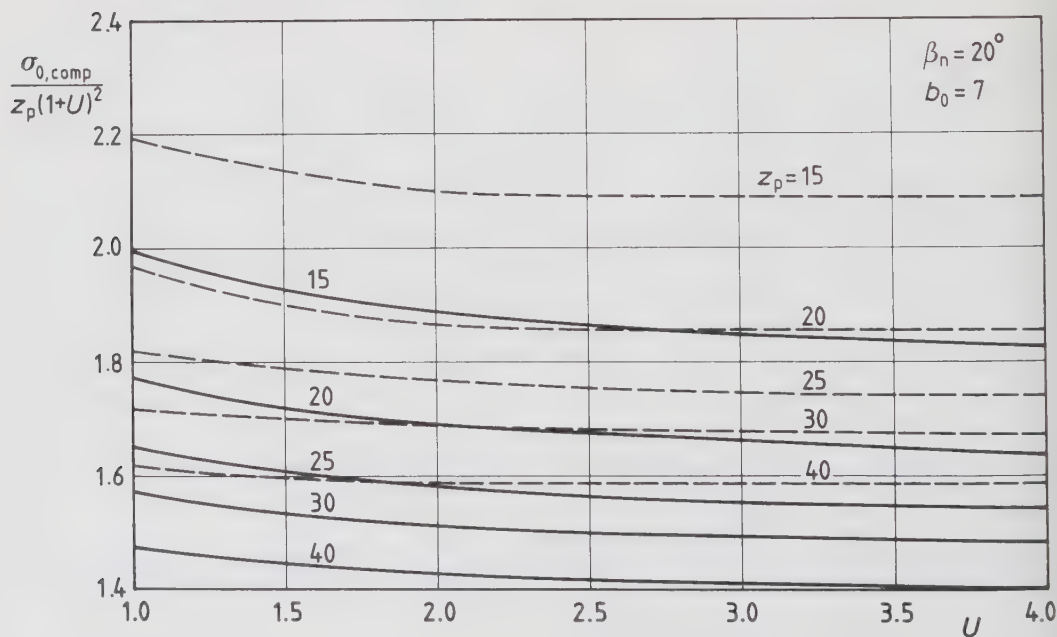


Fig. 6.30. Non-dimensional compression stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

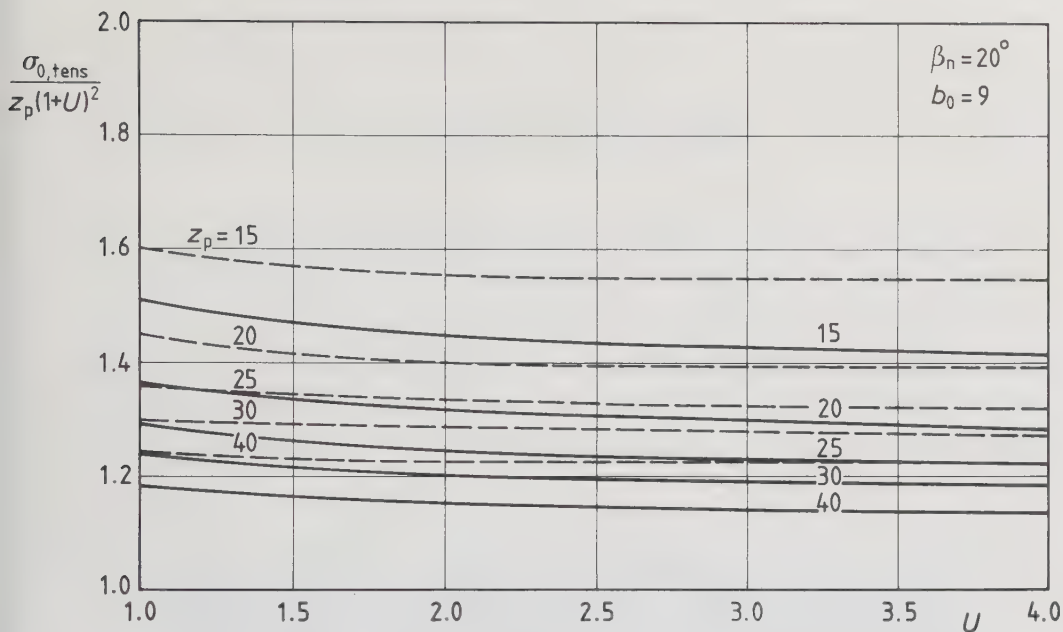


Fig. 6.31. Non-dimensional tensile stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

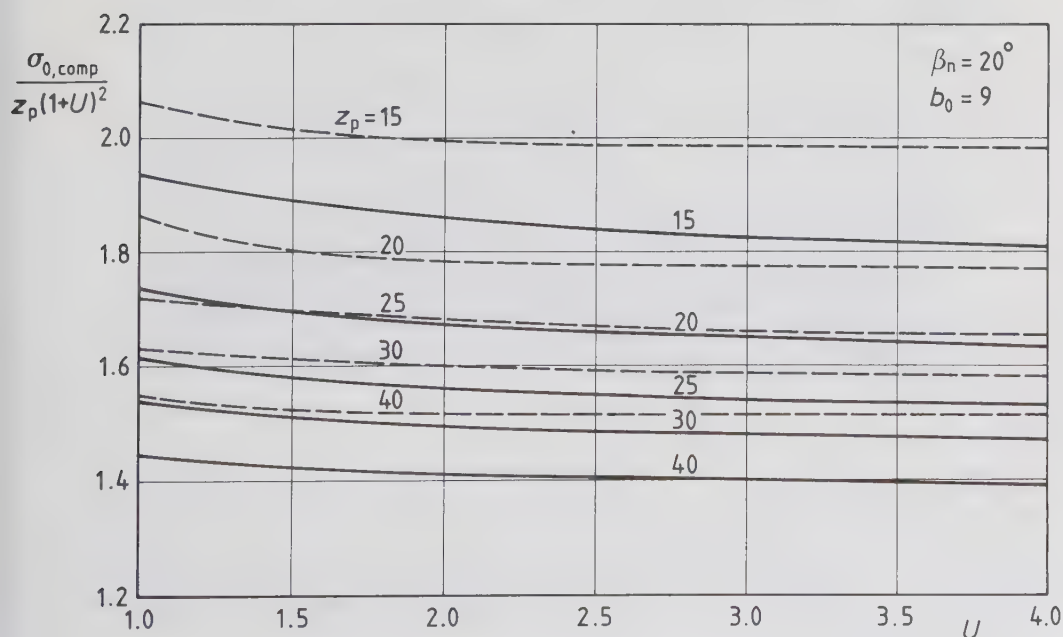


Fig. 6.32. Non-dimensional compression stresses for gears without tip relief (unbroken lines) and for gears with tip relief (dashed lines)

7. Experimental Investigation

In order to verify the theoretical stresses at the fillet a test apparatus was built. The non-modified and unrelieved test gears, made of machine steel SIS 1650, were generated by a tool with $m_n = 6$ mm and $\beta_n = 10^\circ$ and the test gears have the number of teeth $z_p = 20$ and $z_g = 40$ and the face width $b = 30$ mm. The shafts of the gear box were loaded by weights in such a way that equilibrium was reached at all angular positions of the gears where measurements were made.

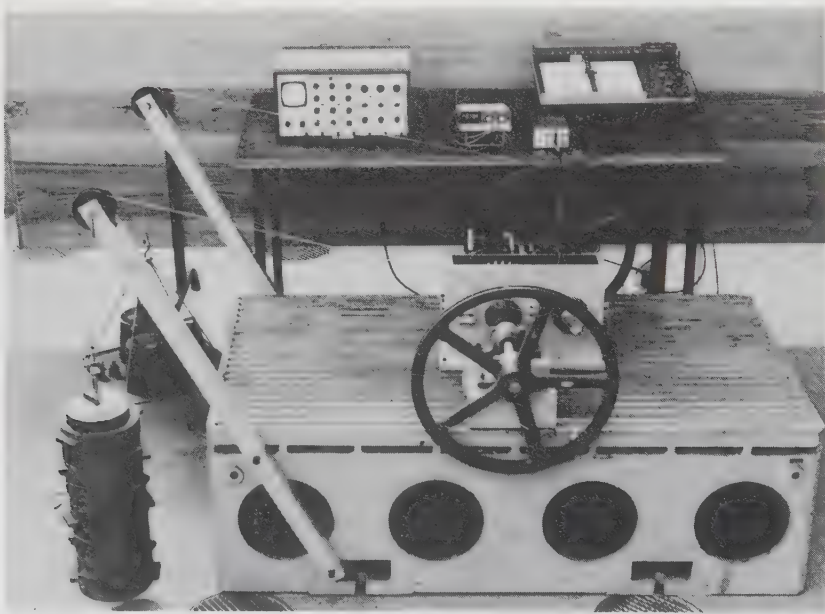


Fig. 7.1. Photo of the test rig

Strain gages with an active length of 0.6 mm were cemented at the fillets of the pinion and the gear at two axial positions at the tensile and compression sides of the measured teeth. The angular position of the pinion, at the current measurement of the strains, were measured with a potentiometer at the non-loaded side of the pinion shaft.

In Figs. 7.2 to 7.5 the results for two test series, where the strain gages were placed on different teeth, are shown and compared with the theoretical values of the strain component, unbroken lines, in the part of the path of contact where the calculations were carried out.

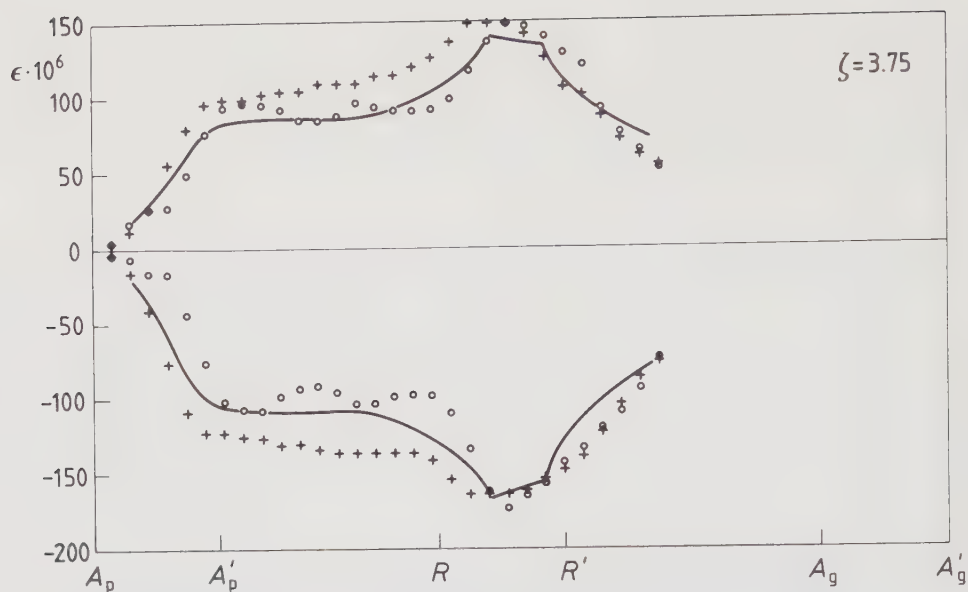


Fig. 7.2. Strain at the tensile and compression sides of the pinion
Theoretical curves and test points

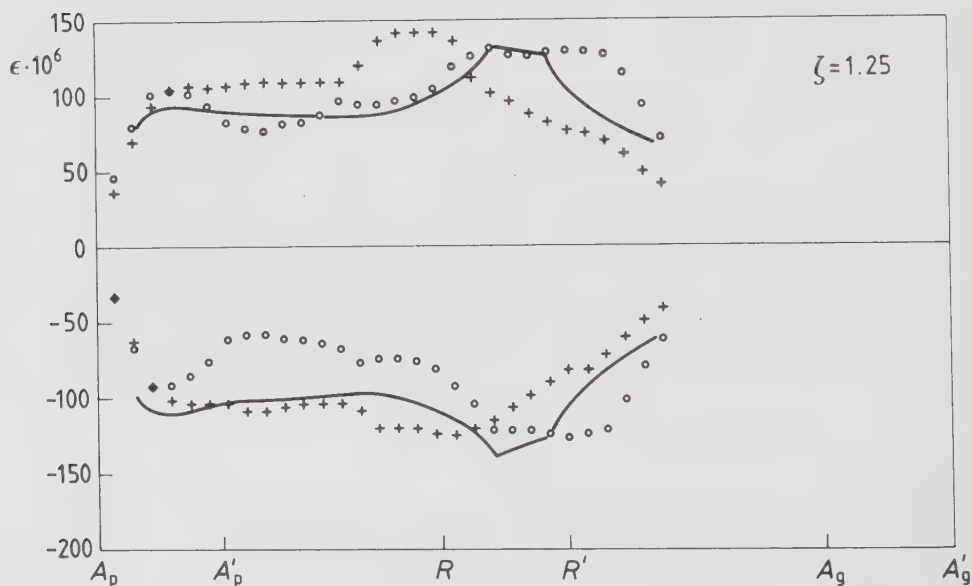


Fig. 7.3. Strain at the tensile and compression sides of the pinion
Theoretical curves and test points

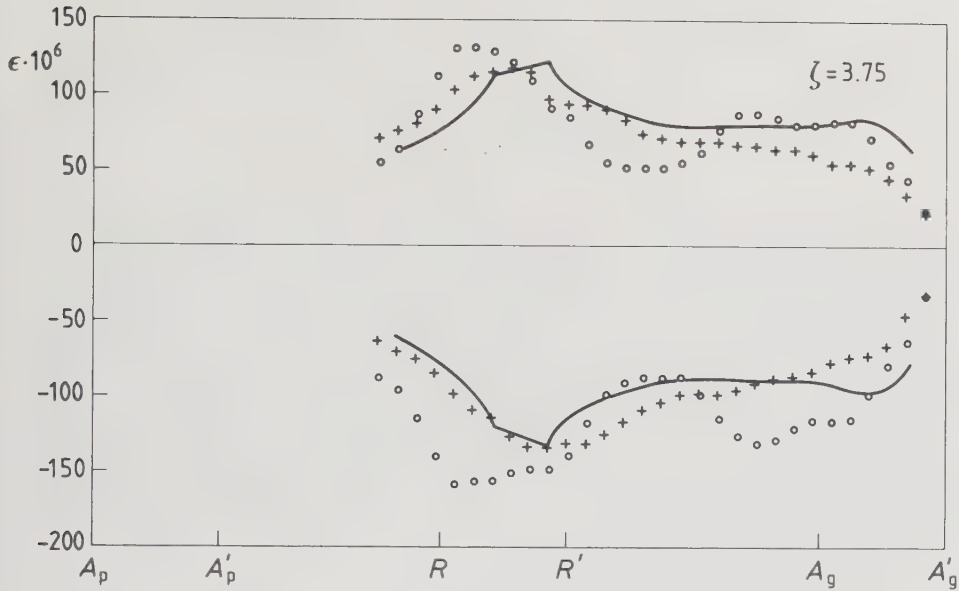


Fig. 7.4. Strain at the tensile and compression sides of the gear
Theoretical curves and test points

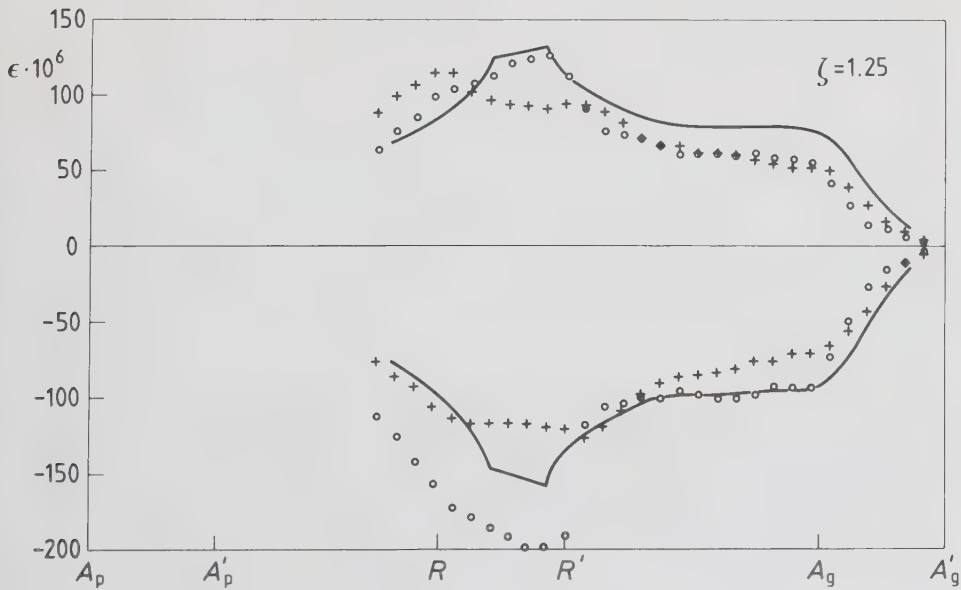


Fig. 7.5. Strain at the tensile and compression sides of the gear
Theoretical curves and test points

8. Conclusion

External involute helical gears are the subject of this work. Geometrical conditions are given. The load distribution between the teeth in contact in the meshing plane were calculated and used in the calculation of the non-dimensional maximum contact pressure and maximum tensile and compression stresses.

An experiment was carried out where the strains at the fillet surface were measured and compared with the calculated strains. The agreement between calculation and test is satisfactory.

9. Appendices

A1. Three-dimensional Study of the Contact Deformation

In section 4.1 an equation was given for the displacement between two rollers due to Hertzian stress distribution where the load intensity q and its extension $2L$ were constants along the contact line. Since this assumption obviously is not true, Eq. 4.1 will be compared with a few cases where the variation of q should be taken into consideration.

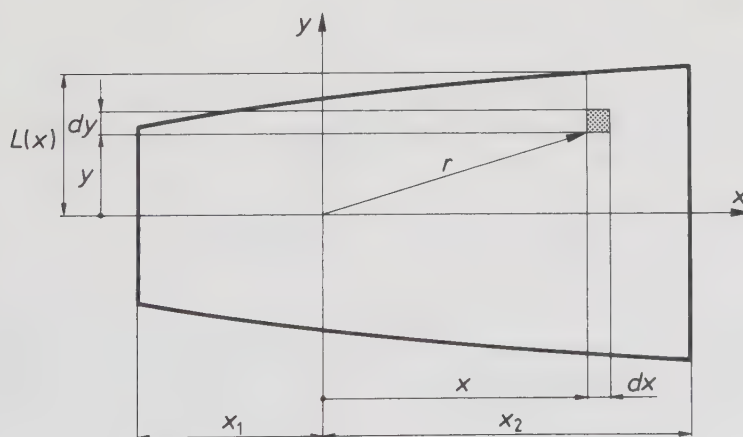


Fig. A1.1. Contact deformation zone

Provided the extension of the load intensity in a certain position at the line of contact is fully determined by the conditions of the contact surfaces and the load intensity in the same position, we have with Hertzian distribution

$$\left\{ \begin{array}{l} \frac{q(x)}{L^2(x)} \quad \rho(x) = \text{constant} \\ q(x) = \frac{\pi L(x) \cdot p(x, 0)}{2} \\ p(x, y) = p(x, 0) \sqrt{1 - \left(\frac{y}{L(x)}\right)^2} \end{array} \right. \quad (\text{A1.1})$$

where the coordinates x and y in the contact zone are shown in Fig. A 1.1.

From these equations we obtain the pressure distribution

$$p(x, y) = \frac{2}{\pi} \frac{q(0)}{L^2(0)} \frac{\rho(0)}{\rho(x)} \sqrt{L^2(x) - y^2} \quad (\text{A1.2})$$

According to Timoshenko (7) the displacement in a semi-infinite solid in a point at the distance z below the surface is due to the loaded surface element $dx dy$

$$u = \frac{p dx dy}{2\pi E} \left[\frac{(1+\nu)z^2}{\sqrt{(r^2 + z^2)^3}} + \frac{2(1-\nu^2)}{\sqrt{r^2 + z^2}} \right]$$

Using the pressure in Eq. A1.2 the displacement between the planes $z = 0$ and $z = h$ in $x = y = 0$ now becomes

$$u = \frac{2}{\pi^2 E} \frac{q(0)}{L^2(0)} \int_{x=-x_1}^{x=x_2} \frac{\rho(0)}{\rho(x)} \int_{y=0}^{y=L(x)} \sqrt{L^2(x) - y^2} \cdot \left[\frac{2(1-\nu^2)}{\sqrt{y^2 + x^2}} - \frac{2(1-\nu^2)}{\sqrt{h^2 + y^2 + x^2}} - \frac{(1+\nu)h^2}{\sqrt{(h^2 + y^2 + x^2)^3}} \right] dy dx$$

Using

$$\begin{cases} x_o = x/m_n \\ y_o = y/m_n \\ h_o = h/m_n \\ L_o = L/m_n \\ \rho_o = \rho/m_n \end{cases}$$

we obtain

$$\begin{aligned} u \frac{E}{q(0)} &= \frac{2}{\pi^2 L_o^2(0)} \int_{x_o = -x_{o1}}^{x_o = x_{o2}} \frac{\rho_o(0)}{\rho_o(x_o)} \int_{y_o = 0}^{y_o = L_o(x_o)} \sqrt{L_o^2(x_o) - y_o^2} \cdot \\ &\cdot \left[\frac{2(1-\nu^2)}{\sqrt{y_o^2 + x_o^2}} - \frac{2(1-\nu^2)}{\sqrt{h_o^2 + y_o^2 + x_o^2}} - \frac{(1+\nu)h_o^2}{\sqrt{(h_o^2 + y_o^2 + x_o^2)^3}} \right] dy_o dx_o \end{aligned} \quad (A1.3)$$

Since $\rho(x)$ affects $q(x)$ in the same way as $L^{-2}(x)$ and to make the comparison a little easier, we set

$$\begin{cases} \rho(x) = \text{constant} \\ L_o(x_o) = L_o(x_{o2}) \left[\frac{L(x_1)}{L(x_2)} + \left(1 - \frac{L(x_1)}{L(x_2)}\right) \frac{x_1 + x}{x_1 + x_2} \right] \end{cases} \quad (A1.4)$$

i.e. the boundaries of the contact area are straight lines corresponding to quadratic variation of the load intensity.

The integrand is singular in the point $x_o = y_o = 0$ but in a small area round this point

$$\begin{aligned} \int_{x_o=0}^{x_o=\epsilon_x} \int_{y_o=0}^{y_o=\epsilon_y} \sqrt{\frac{L_o^2(x_o) - y_o^2}{y_o^2 + x_o^2}} dy_o dx_o &\simeq \\ &\simeq \int_{x_o=0}^{x_o=\epsilon_x} \int_{y_o=0}^{y_o=\epsilon_y} \frac{L_o(0) - \frac{y_o^2}{2L_o(0)}}{\sqrt{y_o^2 + x_o^2}} dy_o dx_o = \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\epsilon_x} \left[L_o(0) \ln(y_o + \sqrt{y_o^2 + x_o^2}) - \frac{y_o}{4L_o(0)} \sqrt{y_o^2 + x_o^2} + \right. \\
&\quad \left. + \frac{x_o^2}{4L_o(0)} \ln(y_o + \sqrt{y_o^2 + x_o^2}) \right]_0^{\epsilon_y} dx_o = \\
&= \int_0^{\epsilon_x} \left[(L_o(0) + \frac{x_o^2}{4L_o(0)}) \operatorname{arcsinh} \frac{\epsilon_y}{x_o} - \frac{\epsilon_y}{4L_o(0)} \sqrt{\epsilon_y^2 + x_o^2} \right] dx_o = \\
&= \left[\epsilon_y \cdot L_o(0) \left(\frac{x_o}{\epsilon_y} \operatorname{arcsinh} \frac{\epsilon_y}{x_o} + \operatorname{arcsinh} \frac{x_o}{\epsilon_y} \right) + \right. \\
&\quad + \frac{\epsilon_y^3}{4L_o(0)} \left(\frac{x_o^3}{3\epsilon_y^3} \operatorname{arcsinh} \frac{\epsilon_y}{x_o} + \frac{x_o}{6\epsilon_y} \sqrt{1 + \left(\frac{x_o}{\epsilon_y} \right)^2} - \frac{1}{6} \operatorname{arcsinh} \frac{x_o}{\epsilon_y} \right) + \\
&\quad \left. - \frac{\epsilon_y^3}{8L_o(0)} \left(\frac{x_o}{\epsilon_y} \sqrt{1 + \left(\frac{x_o}{\epsilon_y} \right)^2} + \operatorname{arcsinh} \frac{x_o}{\epsilon_y} \right) \right]_0^{\epsilon_x} = \\
&= \left[\ln \frac{x_o + \sqrt{x_o^2 + \epsilon_y^2}}{\epsilon_y} \left[\epsilon_y \cdot L_o(0) - \frac{\epsilon_y^3}{6L_o(0)} \right] + \right. \\
&\quad \left. + \ln \frac{\epsilon_y + \sqrt{x_o^2 + \epsilon_y^2}}{x_o} \left[x_o \cdot L_o(0) + \frac{x_o^3}{12L_o(0)} \right] - \frac{\epsilon_y x_o}{12L_o(0)} \sqrt{x_o^2 + \epsilon_y^2} \right]_0^{\epsilon_x} = \\
&= \ln \frac{\epsilon_x + \sqrt{\epsilon_x^2 + \epsilon_y^2}}{\epsilon_y} \left[\epsilon_y L_o(0) - \frac{\epsilon_y^3}{6L_o(0)} \right] + \\
&\quad + \ln \frac{\epsilon_y + \sqrt{\epsilon_x^2 + \epsilon_y^2}}{\epsilon_x} \left[\epsilon_x L_o(0) + \frac{\epsilon_x^3}{12L_o(0)} \right] - \frac{\epsilon_x \epsilon_y}{12L_o(0)} \sqrt{\epsilon_x^2 + \epsilon_y^2}
\end{aligned} \tag{A1.5}$$

Since the limit of this equation is zero when ϵ_x and ϵ_y are approaching zero, the singularity gives no contribution to the displacement.

Eq. A1.5 is used nearest the point where the displacement is to be determined. The rest of the integrand and the integral in the remaining part of the

contact zone are solved numerically by Simpsons formula.

In Fig. A1.2 the displacements are shown for $L_o(x_{o2}) = 0.1$, $h_o = 0.5$ and $x_{o1} + x_{o2} = 5$ for some values of $L(x_1)/L(x_2)$ (unbroken lines) compared with the displacements from Eq. 4.1 (dashed lines).

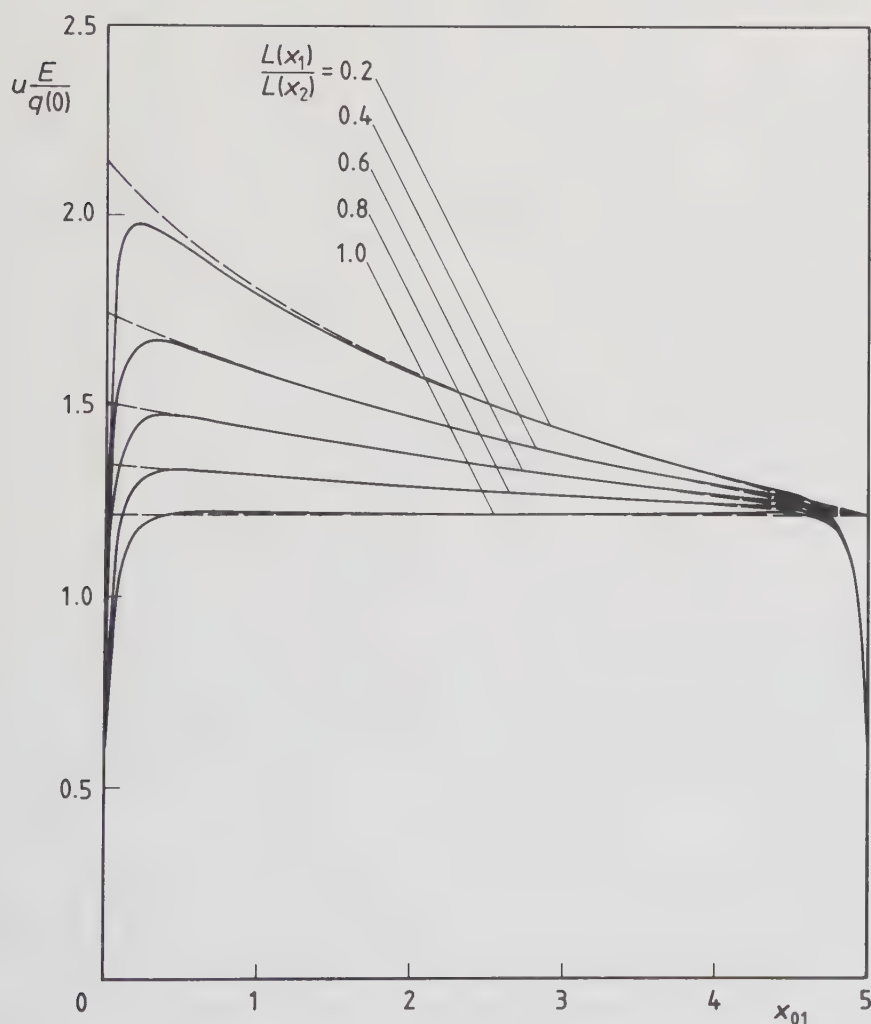


Fig. A1.2. Contact displacements

Obviously the difference between the displacements calculated by the two methods are very small except at the beginning and the end of the contact where the three-dimensional method gives displacements almost half the values

obtained from Eq. 4.1. However no consideration has been taken to the free edges in these points so that the displacements are too low near the points $x_{o1} = 0$ and $x_{o1} = 5$.

The good agreement is of course an effect of the narrow region of the load giving a significant contribution to the displacement. From Fig. A1.2 the extension of this region can be estimated to approximately $\pm 0.25 m_n$.

A2. Influence Function of Bending Deformation

In section 4.2 an approximate equation was given for the bending deformation of the tooth, containing the functions ψ and Γ . For these functions the following formulas are used

$$\left\{ \begin{array}{l} \psi \cdot E \cdot m_n = A_{11} + A_{12} \cdot e^{-A_{13} \cdot \eta} + (B_{11} + B_{12} \cdot e^{-B_{13} \cdot \eta}) \cdot e^{-(C_{11} + C_{12} \eta) \zeta} + \\ \quad + (B_{21} + B_{22} \cdot e^{-B_{23} \cdot \eta}) \cdot e^{-(C_{21} + C_{22}) (b_o - \zeta)} \\ \Gamma = e^{-\left(A_1 + A_2 (\eta + \eta_F) + A_3 \eta \eta_F + A_4 (\eta^2 + \eta_F^2) + A_5 \eta \eta_F (\eta + \eta_F) + A_6 \eta^2 \eta_F^2 \right) |\zeta - \zeta_F|} \end{array} \right. ^B$$

where A , B and C are constants given in Tables A2.1 to A2.3 and η here is the distance from the outermost contact point of the involute flank to the current point where the deformation is to be determined, or where the unit force acts, along the symmetry line of the tooth at the current transverse plane defined by ζ and ζ_F respectively, thus making η independent of ζ at equal radius of the points at the involute flank.

The formula for ψ is dependent on the distance h in Eq. 4.6 and is here set to $0.5 \cdot m_n \cdot \cos \theta$.

When $\beta_n = 0$ the function ψ is symmetric with respect to the plane $\zeta = b_o/2$ for which reason $B_{ij} = B_{ji}$ and $C_{ij} = C_{ji}$. Furthermore, $A_5 = A_6 = 0$ is set to maintain the highest terms quadric, and $A_4 = 0$ since $\eta = \eta_F$ in this case.

The constant B is set to 1.41 when $\beta_n = 0$, 1.38 when $\beta_n = 10^\circ$ and 1.41 when $\beta_n = 20^\circ$.

Table A2.1. Constants in the ψ and Γ formulas for $\beta_n = 0$

Const.	z						
	18	20	25	30	40	80	160
A_{11}	0.8262	0.7568	0.6582	0.6132	0.5725	0.5344	0.5217
A_{12}	2.8553	2.8665	2.8797	2.8783	2.8737	2.8717	2.8791
A_{13}	1.4990	1.4738	1.4372	1.4362	1.4502	1.4970	1.5300
B_{11}	1.2918	1.1604	1.0145	0.9414	0.8784	0.8168	0.7921
B_{12}	4.0629	4.1324	4.1777	4.1962	4.2038	4.2097	4.2206
B_{13}	1.0328	1.0115	1.0164	1.0265	1.0502	1.1003	1.1293
C_{11}	2.1081	2.1251	2.1599	2.1796	2.2026	2.2316	2.2435
C_{12}	0.3675	0.3844	0.4342	0.4680	0.5134	0.5884	0.6283
A_1	0.4449	0.4562	0.4741	0.4858	0.4994	0.5184	0.5273
A_2	-0.0024	-0.0165	-0.0323	-0.0412	-0.0485	-0.0535	-0.0543
A_3	0.0918	0.1211	0.1592	0.1802	0.2003	0.2207	0.2283

Table A2.2. Constants in the ψ and Γ formulas for $\beta_n = 10^\circ$

Const.	z						
	17	20	25	30	40	80	160
A_{11}	0.8498	0.7421	0.6553	0.6139	0.5764	0.5402	0.5276
A_{12}	2.8522	2.8729	2.8784	2.8761	2.8711	2.8700	2.8775
A_{13}	1.5048	1.4565	1.4351	1.4362	1.4518	1.4985	1.5301
B_{11}	1.4757	1.2969	1.1708	1.1139	1.0646	1.0026	0.9674
B_{12}	5.4953	5.5613	5.5803	5.5764	5.5657	5.5681	5.5895
B_{13}	0.8184	0.8181	0.8348	0.8519	0.8794	0.9261	0.9495
C_{11}	2.1147	2.1408	2.1697	2.1876	2.2095	2.2396	2.2530
C_{12}	0.3362	0.3637	0.3950	0.4174	0.4446	0.4855	0.5060
B_{21}	1.0915	0.9267	0.7893	0.7237	0.6643	0.6067	0.5873
B_{22}	3.2000	3.2790	3.3345	3.3551	3.3683	3.3812	3.3909
B_{23}	1.2517	1.2118	1.1964	1.2014	1.2208	1.2696	1.3017
C_{21}	2.1424	2.1773	2.2123	2.2337	2.2550	2.2784	2.2858
C_{22}	0.4458	0.4935	0.5598	0.6116	0.6901	0.8275	0.9075
A_1	0.4294	0.4447	0.4584	0.4688	0.4820	0.5009	0.5109
A_2	0.0406	0.0295	0.0332	0.0356	0.0422	0.0577	0.0605
A_3	-1.9891	-1.8711	-2.0375	-2.0819	-2.1062	-2.2258	-2.3664
A_4	0.8458	0.8574	0.9617	0.9876	0.9892	1.0098	1.0709
A_5	0.2549	0.1484	0.1032	0.0930	0.0994	0.1373	0.1495
A_6	-0.1856	-0.0922	-0.0471	-0.0353	-0.0349	-0.0542	-0.0611

Table A2.3. Constants in the ψ and Γ formulas for $\beta_n = 20^\circ$

Const.	z						
	15	20	25	30	40	80	160
A_{11}	0.8485	0.6973	0.6349	0.6045	0.5757	0.5456	0.5345
A_{12}	2.8908	2.8909	2.8862	2.8805	2.8739	2.8726	2.8791
A_{13}	1.4483	1.4076	1.4049	1.4139	1.4341	1.4804	1.5097
B_{11}	1.0750	0.8710	0.8673	0.8674	0.8601	0.8102	0.7618
B_{12}	8.3345	8.3213	8.2206	8.1624	8.1114	8.1051	8.1413
B_{13}	0.5825	0.6077	0.6370	0.6579	0.6848	0.7232	0.7407
C_{11}	2.1834	2.2275	2.2531	2.2697	2.2901	2.3195	2.3339
C_{12}	0.3316	0.3690	0.3886	0.4007	0.4146	0.4328	0.4403
B_{21}	0.9246	0.6811	0.5848	0.5366	0.4892	0.4422	0.4252
B_{22}	2.6299	2.7492	2.7842	2.7988	2.8125	2.8277	2.8373
B_{23}	1.5412	1.4278	1.4123	1.4164	1.4318	1.4782	1.5081
C_{21}	2.1865	2.2590	2.2888	2.3030	2.3135	2.3012	2.2759
C_{22}	0.6114	0.7812	0.9081	1.0153	1.1778	1.5190	1.7631
A_1	0.4602	0.4861	0.4987	0.5178	0.5310	0.5497	0.5639
A_2	0.0074	-0.0146	-0.0023	-0.0388	-0.0343	-0.0244	-0.0243
A_3	-0.3276	-0.1784	-0.1769	0.0550	0.1353	0.2305	0.4984
A_4	0.1200	0.1511	0.1354	0.1253	0.0795	0.0059	-0.1161
A_5	0.1474	0.0016	0.0025	-0.0965	-0.0910	-0.0595	-0.0749
A_6	-0.1160	0.0080	0.0177	0.0758	0.0737	0.0543	0.0648

A3. Tables of Calculated Values

Table A3.1. Non-dimensional spur gear contact pressures

z_g	$b_0 (\delta_1 = 0)$		δ_1	2-dim.	$b_0 (\delta_1 \neq 0)$	
	5	15			5	15
$z_p = 18$						
18	2.96	3.00	0	2.89	2.96	3.00
20	3.08	3.11	0	3.00	3.08	3.11
25	3.39	3.44	0.001	3.29	3.37	3.41
30	3.91	3.96	0.001	3.58	3.67	3.72
40	5.00	5.06	0.002	4.18	4.29	4.34
80	9.85	9.96	0.003	6.65	6.82	6.91
$z_p = 20$						
20	2.94	2.97	0	2.87	2.94	2.97
25	3.19	3.23	0	3.11	3.19	3.23
30	3.45	3.49	0	3.36	3.45	3.49
40	3.98	4.03	0	3.89	3.98	4.03
80	7.02	7.10	0.001	6.04	6.20	6.28
$z_p = 25$						
25	2.91	2.95	0	2.84	2.91	2.95
30	3.10	3.14	0	3.02	3.10	3.14
40	3.50	3.54	0	3.41	3.50	3.54
80	5.18	5.24	0	5.04	5.18	5.24
160	8.67	8.76	0	8.42	8.67	8.76
$z_p = 30$						
30	2.90	2.94	0	2.83	2.90	2.94
40	3.21	3.25	0	3.12	3.21	3.25
80	4.56	4.61	0	4.43	4.56	4.61
160	7.39	7.48	0	7.17	7.39	7.48
$z_p = 40$						
40	2.90	2.94	0	2.82	2.90	2.94
80	3.84	3.88	0	3.72	3.84	3.88
160	5.90	5.96	0	5.70	5.90	5.96

Table A3.2. Non-dimensional helical gear contact pressures for $\beta_n = 10^\circ$

z_g	$b_0 (\delta_1 = 0)$			δ_1	$b_0 (\delta_1 \neq 0)$		
	5	7	9		5	7	9
$z_p = 17$							
17	6.38	6.41	6.44	0.007	3.06	3.10	3.12
20	7.02	7.06	7.11	0.006	3.25	3.30	3.32
25	8.17	8.23	7.97	0.005	3.61	3.66	3.68
30	9.38	9.45	9.01	0.005	4.00	4.06	4.09
40	11.95	12.04	11.26	0.004	4.81	4.87	4.92
80	23.45	22.76	21.40	0.003	8.25	8.33	8.40
$z_p = 20$							
20	5.69	5.73	5.58	0.007	2.88	2.88	2.87
25	6.39	6.43	6.14	0.006	3.11	3.11	3.08
30	7.11	7.17	6.74	0.006	3.35	3.36	3.37
40	8.61	8.68	8.02	0.006	3.87	3.92	3.97
80	14.83	14.10	13.40	0.005	6.13	6.25	6.34
$z_p = 25$							
25	5.22	5.25	4.93	0.006	2.87	2.87	2.80
30	5.67	5.72	5.29	0.006	3.04	3.05	2.96
40	6.62	6.48	6.07	0.006	3.42	3.41	3.31
80	10.52	9.78	9.37	0.006	5.05	5.02	4.87
160	18.47	16.73	16.15	0.006	8.43	8.35	8.27
$z_p = 30$							
30	5.03	5.00	4.63	0.006	2.87	2.86	2.76
40	5.72	5.48	5.19	0.006	3.15	3.12	3.02
80	8.66	7.92	7.63	0.006	4.46	4.37	4.23
160	14.20	13.10	12.69	0.006	7.21	7.03	6.82
$z_p = 40$							
40	4.90	4.57	4.38	0.006	2.87	2.79	2.69
80	6.77	6.19	5.99	0.007	3.77	3.65	3.54
160	10.22	9.69	9.43	0.007	5.72	5.54	5.39

Table A3.3. Non-dimensional helical gear contact pressures for $\beta_n = 20^\circ$

z_g	$b_0 (\delta_1 = 0)$			δ_1	$b_0 (\delta_1 \neq 0)$		
	5	7	9		5	7	9
$z_p = 15$							
15	6.29	5.84	5.60	0.005	3.47	3.50	3.34
20	7.52	6.88	6.62	0.004	3.91	3.88	3.67
25	8.84	8.01	7.73	0.004	4.40	4.37	4.15
30	10.23	9.19	8.90	0.003	4.90	4.75	4.53
40	12.86	11.71	11.35	0.003	5.94	5.74	5.49
80	24.42	22.85	22.24	0.003	10.03	9.66	9.20
$z_p = 20$							
20	5.31	4.80	4.64	0.006	3.01	3.02	2.88
25	5.92	5.30	5.13	0.005	3.22	3.19	3.05
30	6.33	5.82	5.64	0.005	3.52	3.48	3.33
40	7.39	6.89	6.70	0.004	4.04	3.91	3.76
80	11.95	11.31	11.04	0.004	6.47	6.22	5.98
$z_p = 25$							
25	4.82	4.45	4.32	0.006	2.78	2.75	2.64
30	5.14	4.79	4.66	0.006	2.98	2.94	2.82
40	5.85	5.50	5.36	0.006	3.40	3.35	3.21
80	8.92	8.50	8.31	0.005	5.04	4.85	4.68
160	15.25	14.62	14.33	0.005	8.56	8.20	7.91
$z_p = 30$							
30	4.57	4.29	4.18	0.006	2.71	2.60	2.50
40	5.10	4.83	4.71	0.006	2.96	2.89	2.79
80	7.45	7.13	6.98	0.006	4.28	4.14	4.01
160	12.34	11.87	11.66	0.006	7.03	6.73	6.56
$z_p = 40$							
40	4.38	4.17	4.08	0.007	2.67	2.51	2.47
80	5.98	5.75	5.65	0.007	3.47	3.33	3.27
160	9.41	9.10	8.97	0.008	5.42	5.22	5.12

Table A3.4. Non-dimensional spur gear fillet stresses

z_g	$\sigma_{0, \text{tens}}$	$\sigma_{0, \text{comp}}$
$z_p = 18$		
18	116	133
20	128	147
25	162	186
30	201	228
40	289	327
80	807	904
$z_p = 20$		
20	124	142
25	155	177
30	190	216
40	270	305
80	732	820
$z_p = 25$		
25	146	167
30	174	199
40	240	273
80	612	687
160	1871	2084
$z_p = 30$		
30	168	191
40	225	255
80	540	607
160	1585	1770
$z_p = 40$		
40	212	240
80	465	520
160	1269	1412

Table A3.5. Non-dimensional helical gear tensile stresses for $\beta_n = 10^\circ$

z_g	$b_0 (\delta_1 = 0)$			$b_0 (\delta_1 \neq 0)$		
	5	7	9	5	7	9
$z_p = 17$						
17	110	110	110	110	110	110
20	129	129	127	129	129	129
25	165	165	154	165	165	165
30	206	206	188	206	206	205
40	300	300	266	300	300	298
80	858	803	727	858	858	843
$z_p = 20$						
20	122	122	115	122	122	122
25	154	154	140	154	154	152
30	188	188	168	188	188	187
40	269	269	234	269	269	269
80	738	664	614	738	741	729
$z_p = 25$						
25	144	144	127	144	144	139
30	173	173	151	173	173	167
40	240	226	204	240	238	231
80	619	534	502	619	615	593
160	1906	1582	1504	1906	1890	1825
$z_p = 30$						
30	166	160	142	166	165	158
40	224	203	188	224	220	212
80	547	461	438	547	531	511
160	1520	1314	1263	1620	1562	1500
$z_p = 40$						
40	212	185	175	212	204	196
80	449	385	373	469	452	435
160	1122	1026	1001	1277	1226	1182

Table A3.6. Non-dimensional helical gear compression stresses for $\beta_n = 10^\circ$

z_g	$b_0 (\delta_1 = 0)$			$b_0 (\delta_1 \neq 0)$		
	5	7	9	5	7	9
$z_p = 17$						
17	134	134	134	134	134	134
20	158	158	155	158	158	158
25	200	200	189	200	200	201
30	248	248	230	248	248	251
40	361	361	326	361	361	365
80	1020	960	889	1020	1025	1031
$z_p = 20$						
20	150	150	141	150	150	150
25	187	187	172	187	187	187
30	229	229	207	229	229	230
40	325	325	288	325	325	327
80	878	799	754	878	892	896
$z_p = 25$						
25	177	177	159	177	177	173
30	211	211	187	211	211	208
40	291	275	253	291	290	288
80	738	651	624	738	748	738
160	2247	1926	1872	2247	2301	2271
$z_p = 30$						
30	203	197	178	203	202	197
40	273	249	235	273	270	265
80	654	565	548	654	652	638
160	1799	1611	1579	1918	1915	1874
$z_p = 40$						
40	259	228	218	259	251	244
80	541	476	464	566	557	542
160	1353	1266	1248	1539	1512	1473

Table A3.7. Non-dimensional helical gear tensile stresses for $\beta_n = 20^\circ$

z_g	$b_0 (\delta_1 = 0)$			$b_0 (\delta_1 \neq 0)$		
	5	7	9	5	7	9
$z_p = 15$						
15	109	93	90	109	103	96
20	147	124	121	147	137	129
25	192	159	156	192	176	167
30	242	199	196	242	221	210
40	342	291	288	360	329	312
80	950	843	840	1072	978	929
$z_p = 20$						
20	133	109	109	133	122	116
25	164	136	136	168	150	145
30	193	165	167	205	182	178
40	266	234	237	288	259	252
80	698	630	642	795	714	696
$z_p = 25$						
25	145	128	129	155	140	136
30	171	153	154	186	169	164
40	231	209	213	258	234	227
80	571	529	541	653	595	583
160	1720	1613	1657	2015	1834	1801
$z_p = 30$						
30	160	147	149	175	160	156
40	211	196	200	235	216	211
80	495	470	481	570	527	517
160	1435	1375	1416	1684	1557	1519
$z_p = 40$						
40	196	185	189	221	203	199
80	420	404	415	483	449	441
160	1136	1103	1137	1346	1250	1226

Table A3.8. Non-dimensional helical gear compression stresses for $\beta_n = 20^\circ$

z_g	$b_0 (\delta_1 = 0)$			$b_0 (\delta_1 \neq 0)$		
	5	7	9	5	7	9
$z_p = 15$						
15	136	120	116	136	131	124
20	182	159	155	182	175	165
25	236	204	200	236	226	214
30	297	255	251	297	283	269
40	419	374	370	440	422	400
80	1162	1082	1077	1312	1255	1192
$z_p = 20$						
20	165	142	139	165	157	149
25	201	177	173	206	195	185
30	237	215	212	252	238	225
40	327	304	301	354	336	320
80	858	818	817	977	927	885
$z_p = 25$						
25	182	165	161	194	182	172
30	214	197	193	233	219	207
40	289	271	266	324	301	286
80	715	684	678	817	769	731
160	2154	2085	2076	2523	2372	2258
$z_p = 30$						
30	203	188	185	221	206	195
40	267	252	248	298	278	264
80	627	604	597	722	677	642
160	1818	1768	1757	2134	2001	1897
$z_p = 40$						
40	249	235	231	280	259	248
80	533	514	508	613	570	543
160	1442	1402	1390	1709	1588	1517

10. References

1. Andersson, S.A.E.: On the Design of Internal Involute Spur Gears. Trans. of Machine Elements Division, Lund Technical University, Lund 1973.
2. Buckingham, E.: Analytical Mechanics of Gears. McGraw-Hill, New York 1949.
3. Hayashi, K. & Sayama, T.: Load Distribution on the Contact Line of Helical Gear Teeth. Bulletin of JSME 1961.
4. Hernelind, J. & Pärletun, L.G.: EUFEMI, A Finite Element Computer Program for Solid Mechanics and Heat Conduction. Lund Technical University, Division of Solid Mechanics 1976.
5. Lewis, W.: Investigation of the Strength of Gear Teeth. Proceedings of Engineers' Club of Philadelphia 1893.
6. Niemann, G.: Maschinenelemente. Band 2. Berlin (Springer) 1961.
7. Timoshenko, S. & Goodier, J.N.: Theory of Elasticity. McGraw-Hill, New York 1970.
8. Weber, C. & Banaschek, K.: Schriftenreihe Antriebstechnik, Ht 11. Friedr. Vieweg & Sohn, Braunschweig 1953.
9. Wennerström, E.: On the Optimum Design of External Involute Gears. Trans. of Machine Elements Division, Lund Technical University, Lund 1973.
10. Wennerström, E.: Stresses in External Involute Spur Gears. Acta Polytechnica Scandinavica, Mechanical Engineering Series No. 53, 1970.

Transactions of
MACHINE ELEMENTS DIVISION
Lund Technical University, Lund, Sweden

- Gerbert, B G: Power Loss and Optimum Tensioning of V-belt Drives, 1972.
- Gerbert, B G: Tensile Stress Distribution in the Cord of V-belts, 1972.
- Andersson, B: Calculation of Contact Force for Hydrodynamic Lip Seals Including Optimization with Regard to Cross Section of the Lip, 1973.
- Andersson, S A E: On the Design of Internal Involute Spur Gears, 1973.
- Floberg, L: Experimental Investigation of Pure Wear Under the Condition of Boundary Lubrication, 1973.
- Floberg, L: On Journal Bearing Lubrication Considering the Tensile Strength of the Liquid Lubricant, 1973.
- Floberg, L: On the Tensile Strength of Liquids, 1973.
- Gerbert, B G: Scheibenspreizkräfte in Breitkeilriemengetrieben, 1973.
- Gerbert, B G: Pressure Distribution and Belt Deformation in V-belt Drives, 1973.
- Jacobson, B: Elasto-solidifying Lubrication of Spherical Surfaces, 1973.
- Jacobson, B: On the Starved Lubrication of Cylindrical Rolling Contacts, Considering Solidification of the Lubricant, 1973.
- Lundholm, G: On Whirl Frequencies and Stability Borderlines for Journal Bearings, 1973.
- Wennerström, E: On the Optimum Design of External Involute Gears, 1973.
- Wennerström, E: Tension of a Strip with Two Semi-circular Notches, 1973.
- Dike, G: Stresses and Cracks in Brake Discs, 1976.
- Dike, G: Temperature Distribution in Disc Brakes, 1976.
- Gerbert, B G: Non-sliding and Sliding Zones in V-belt Drives, 1976.
- Jansson, L: Forced Whirling of a Rotor, 1976.
- Wallin, A: On Vibrating Hammers, 1978.
- Dike, G: Geometry and Stresses of Straight Bevel Gears, 1978.

